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Frege, Indispensability, and the Compatibilist Heresy†

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ABSTRACT

In *Grundgesetze*, Vol. II, §91, Frege argues that 'it is applicability alone which elevates arithmetic from a game to the rank of a science'. Many view this as an *in nuce* statement of the indispensability argument (IA) later championed by Quine. Garavaso has questioned this attribution. I argue that even though Frege's applicability argument is not a version of IA, it facilitates acceptance of suitable formulations of IA. The prospects for making the empiricist IA compatible with a rationalist Fregean framework appear thus much less dim than expected. Nonetheless, those arguing for such compatibility eventually face an hardly surmountable dilemma.

1. FREGE: INSPIRER OF TWO RIVAL TRADITIONS?

Among contemporary versions of mathematical platonism we can point to two outstanding representatives of the old-fashioned divide between rationalists and empiricists. On the one side, neo-logicists maintain, roughly speaking, that the existence of mathematical objects (the finite cardinals, at least) is warranted *a priori* through a combination of logical knowledge and definitions. On the other side, supporters of (some version of) the so-called indispensability argument (henceforth, IA) maintain, roughly speaking, that the existence of mathematical objects (of whatever kind) is justified *a posteriori* by the role mathematical theories have in true or well-confirmed scientific theories.

Each tradition has its own heroes. In the cases at hand, these are Frege and Quine, respectively. The neo-logicists' aim is to revive Frege's original logicist project, by abandoning the inconsistent Basic Law V and replacing it with the principle today known

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as *Hume's Principle* (henceforth HP),[1] in order to obtain, via appropriate definitions, a derivation of Peano Axioms for arithmetic in a system of (full impredicative) second-order logic with the sole addition of HP as a non-logical axiom. Indispensabilists — so we shall call them — look back at Quine's works for various suggestions leading to a version of IA, and, derivatively, to Putnam's first explicit formulation of it, itself pointing back to Quine's views.[2]

As Garavaso [2005] has already noticed, however, it is very common to find discussions of IA suggesting that an early, if not the earliest, statement of IA can be found in Frege himself.[3] In particular, many read §91 of the second volume of *Die Grundgesetze der Arithmetik* as suggesting some version of IA or, at least, the basic idea underlying it. One should not underestimate this analogy: the textual evidence that traces the ancestry of IA back to Frege's argument in *Grundgesetze* is impressively pervasive.[4] Nonetheless, Garavaso [2005] rejects such an attribution to Frege, and contrasts what we shall call Frege's 'Applicability Argument' (henceforth, AA) with the much-discussed version of IA suggested by [Colyvan, 1998; 2001]. According to Garavaso, Frege is just arguing 'for the thesis that in order to provide an accurate account of mathematics, we need to account for its applicability' (p. 171). At most, the two arguments support different realist theses — that there exist mathematical entities, in Quine's, Putnam's and Colyvan's case; and that mathematical statements express thoughts, in Frege's case. She thus concludes (p. 172):

> … if these are comparable forms of realism, an argument is due to support this analogy; so far, neither Colyvan nor anyone else who claims that Frege has advanced an indispensability argument has yet given such an argument. I do not deny that it might be possible to reconstruct a Fregean indispensability argument

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[1] The second-order principle according to which for every concept *F* and every concept *G*, the number of *F*s is equal to the number of *G*s if and only if the *F*s and the *G*s can be put into one-one correspondence.

[2] The first explicit formulation of IA is in [Putnam, 1971, p. 347]:

> So far I have been developing an argument for realism roughly along the following lines: quantification over mathematical entities is indispensable for science, both formal and physical, therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question.

[3] Another early statement of IA is often spotted in [Gödel, 1947].

[4] A surely incomplete list includes at least [Field, 1984, pp. 94–95, and fn. 17; 1988, p. 142; Maddy, 1992, p. 275; Balaguer, 1998, p. 96; Colyvan, 2001, pp. 8–9; 2007, p. 109; Dieveney, 2007, p. 106, fn. 1; Paseau, 2007, p. 128, fn. 10]. Here is a representative example [Colyvan, 2001, pp. 8–9]:

> The use of indispensability arguments for defending mathematical realism is usually associated with Quine … and Putnam … but it's important to realise that the argument goes back much further. … As Michael Dummett points out [1991, p. 60], Frege's appeal to the applications of arithmetic [in *Grundesetze*, II, §91] is made in order to raise a problem for formalists who liken mathematics to a game in which mathematical symbols have no meaning, but are simply manipulated in accordance with certain rules. Frege asks the formalists to explain how such a game could have applications… . This is clearly a form of indispensability argument.

on the basis of other texts by Frege, but only that Frege does not present one such argument in section 91.

My aims in what follows will be

(a) to suggest a more nuanced reading of Frege's Applicability Argument in *Grundgesetze*, II, §91 than the one offered by Garavaso (Section 2);
(b) to show that, even though Frege does not offer any proper formulation of IA, it is possible, on the basis of what Frege says *in that very section*, to find support for an appropriately formulated version of IA that would be acceptable in a Fregean framework; and
(c) to explore how far someone could go who wished to claim the compatibility of some version of IA with a Fregean point of view, in order to support 'comparable' forms of realism (Section 3).

I will thus suggest that there is much in the basic ideas underlying IA (or at least some version of it) that might be shared by both a Fregean and a Quinean point of view. Hence, the possibility of retrieving from Frege's views some basic ideas that, with suitable provisos, may lend support to a form of IA is less dim than may appear at first. Nonetheless, I ultimately point to an hardly surmountable dilemma that such compatibilist position will eventually face (Section 4). These conclusions have relevance not just for the local debate on IA, but also for a more perspicuous assessment of the relations between the major representatives of the rationalist and the (non-nominalist) empiricist tradition in the philosophy of mathematics.

2. THE INDISPENSABILITY ARGUMENT AND ITS ALLEGED FREGEAN ORIGINS

2.1. The Indispensability Argument and its Main Features

Aside from Quine's scattered remarks and Putnam's original formulation of IA (*cf.* fn. 2), the most discussed version of the argument is the one suggested by [Colyvan, 1998; 2001]:[5]

Colyvan's Indispensability Argument (CIA)

1. We ought to have ontological commitment to all and only those entities that are indispensable to our best scientific theories.
ii. Mathematical entities are indispensable to our best scientific theories.

(CIA) ________________________

iii. We ought to have ontological commitment to mathematical entities.

[5] [Colyvan, 2001, Chap. 1] presents at least three versions of IA (and discusses others, among which the *Pragmatic Indispensability Argument* offered by Resnik [1995]): these he calls the *Scientific Indispensability Argument*, the *Quine/Putnam Indispensability Argument*, and the *Semantic Indispensability Argument*. I will focus on the second, *i.e.*, CIA, since it has been by far the most discussed in the subsequent debate.

Also expanding on the discussion of [Garavaso, 2005, p. 165], we can single out CIA's main features. It will be said to be:

a. INDISPENSABILITY-BASED: it is based on the *indispensability* of mathematics for the sciences;
b. PLATONIST: its main thrust is to support the legitimacy and even the desirability of accepting *the existence of mathematical entities*;
c. ABDUCTIVE: it is ultimately based on an *abductive* (hence, broadly speaking, *inductive*[6]) *inference* (analogously to other arguments for the existence of theoretical entities in physics);
d. NATURALIST: it relies on naturalism;
e. HOLIST: it relies on confirmational holism.

Features (d) and (e) are suggested by [Colyvan, 2001, p. 12] when he claims that 'the crucial first premise follows from the doctrines of *naturalism* and *holism*'.[7] Both doctrines may receive different characterizations, but in this context we can rest content with the following. NATURALISM is the claim that scientific theories are the only source of genuine knowledge. As a consequence, with respect to ontology, we are justified in acknowledging the existence *only* of those entities that are quantified over in our true or well-confirmed scientific theories. CONFIRMATIONAL HOLISM is the claim that empirical evidence does not confirm scientific hypotheses in isolation, but rather scientific theories as a whole. As a consequence, with respect to ontology, we are justified in acknowledging the existence of *all* those entities that are quantified over in our true or well-confirmed scientific theories.

Garavaso's main claim (*ibid.*) is that 'Frege's alleged argument from indispensability differs from the above [argument] in lacking these … features'.[8] Let us then look more closely at Frege's argument.

[6]This proviso can be questioned if abductive inferences are excluded from inductive ones, but I will adopt the same broad classification adopted by Garavaso. Notice that Colyvan's (and others') versions of IA are deductive in form; as [Garavaso, 2005, p. 165] also makes clear, they are based on an abductive inference in the sense that 'an inference to the best explanation [IBE] supports [their] main premise'. Much of recent discussion focuses on the relations between IBE and IA, though it is seldom claimed that IA may rely on enumerative induction. However, Garavaso (p. 165, fn. 7) reports Resnik as commenting that 'these arguments may be regarded as inductive also because the indispensability of mathematics has been established by many failed efforts of talented logicians to dispense with it'. Something similar is suggested by Dummett [1993, p. 433]: 'Frege had never given any good reason for insisting on the genuine existence of mathematical objects; perhaps the only plausible reason lies in the difficulties encountered by Russell and Whitehead in trying to dispense with them'.

[7]In particular, the "only" direction is supposed to follow from naturalism, and the "all" direction is supposed to follow from confirmational holism.

[8]Garavaso considers only features (a) to (c) as being those common to all indispensability arguments discussed by Colyvan (*cf.* fn. 5). She also acknowledges (p. 162) that CIA's first premise 'relies on the acceptance of two Quinean doctrines such as holism and naturalism. Accordingly, [it] cannot be appealing to philosophers unwilling to accept Quine's assumptions.' I will come back to this point in Section 3.1.

2.2. Frege's Applicability Argument(s)

Frege's argument is found in *Grundgesetze* [Frege, 1893–1903, Vol. II, §91] which belongs to a longer part of *Grundgesetze* (§§86–137) where Frege gives his criticisms of the formalists.[9] In Frege's eyes, in claiming that arithmetic consists in just a manipulation of signs regardless of their possessing any sense, formalists are guilty of equating arithmetic with games like chess, where each move is permitted by arbitrarily chosen rules. Frege raises several objections to the formalist conception. What interests us here, is that §91 shows that the possibility of adequately accounting for the *applicability* of arithmetic plays a crucial role in dismissing formalism as a viable position. It is worth quoting the relevant passage in full:

> Whereas in meaningful arithmetic equations and inequations are sentences expressing thoughts, in formal arithmetic they are comparable with the positions of chess pieces, transformed in accordance with certain rules without considerations for any sense. For if they were viewed as having sense, the rules could not be arbitrarily stipulated; they would have to be so chosen that from formulas expressing true propositions could be derived only formulas likewise expressing true propositions. Then the standpoint of formal arithmetic would have to be abandoned, which insists that the rules for the manipulation of signs are quite arbitrarily stipulated. Only subsequently may one ask whether the signs can be given a sense compatible with the rules previously laid down. Such matters, however, lie entirely outside formal arithmetic and only arise when applications are to be made. Then, however, they must be considered; for an arithmetic with no thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it. Is it good, then, to exclude from arithmetic what it needs in order to be a science?

Frege is in this passage concerned with what Steiner [1998] has called the problem of the "semantic applicability" of arithmetic (mathematics),[10] *i.e.*, the problem of how can it be that arithmetical statements — when interpreted at face value, as featuring terms purporting to refer to (abstract) arithmetical objects — can 'function as premises in deductions, including those which predict observations' (p. 16). Why should Frege

[9] More specifically, to the views of J. Thomae and H. Heine. See [Dummett, 1991, ch. 20] for a survey of Frege's discussion of formalism.

[10] As opposed to other related problems of applicability, *e.g.*, metaphysical, descriptive, epistemic … . See also [Steiner, 1995; 2005], and, for a recent discussion of applicability, [Pincock, 2012].

require that an explanation of (semantic) applicability requires that arithmetical statements 'express thoughts', and why should this explanation be precluded to formalists? As Dummett [1991, pp. 256–257] makes clear, this is because

> … he takes the application of a mathematical theorem to be an instance of deductive inference. It is possible to make an inference only from a thought (only from a true thought, that is, from a fact, according to Frege): it would be senseless to infer from something that neither was a thought nor expressed one.

According to [Garavaso, 2005, pp. 171], Frege here 'is not arguing for the truth of mathematical statements or the existence of mathematical entities … . Hence, Frege's applicability argument does not share the second feature of Colyvan's indispensability arguments [*i.e.*, PLATONIST].' Now, one can hardly deny that one of Frege's aims in the relevant passage is to show that a proper account of mathematics includes an explanation of applicability; that the obvious fact that mathematics is applicable requires mathematics to possess whatever features are necessary for its applicability; and that formalists cannot account for these features. But one should not see this as the whole point of Frege's discussion. Rather, one can look at §91 as conflating distinct arguments with distinct conclusions, all together pointing to the inadequacies of the formalist account. Frege is attacking two distinct features of formalism: that the rules of arithmetic (that which regulates transitions from some formulae to others) are arbitrarily stipulated like those of chess or games in general; and that arithmetical formulae do not express thoughts. The arguments in §91 concern these two claims.

One first argument goes against the claim that arithmetic's rules are arbitrary. Let "x" vary over everything that has rules which allow the transition from one or more configurations of signs — which can be either linguistic signs or chess pieces — to other configurations of signs. Now, as Frege argues, x's rules can be arbitrary only if x is a game, and if x is a game, it cannot be applied (or so Frege suggests). But arithmetic can be applied (it is indeed applied); thus arithmetic is not a game, and therefore its rules cannot be arbitrary.

A second argument, once Dummett's suggestions above are taken into account, goes against the claim that arithmetical formulae express no thoughts. Let "CoS" stand for "configuration of signs", intended as above. An application of a CoS of x is an instance of a sound deductive inference in which that particular CoS occurs, and a CoS of x can occur in a sound deductive inference only if it expresses a thought. Since the premises of a sound deductive inference whose conclusion is true must also be true, it follows that a CoS of x can occur in a sound deductive inference only if it expresses a true thought. If a CoS of x can be applied, then, it must be possible for it to occur in a deductive sound inference, and thus it must express a true thought. But arithmetical formulae (*i.e.*, arithmetic's configurations of signs) can be applied (are indeed applied). Therefore they express true thoughts and, *a fortiori*, they express thoughts.

We have here two different though related arguments, their connection being that the reason why only games could have arbitrary rules is that if configurations of signs expressed thoughts — were meaningful — transitions among them would be constrained by the content of the thoughts they express. Both arguments hinge on the role played by the applicability of arithmetic. Let us call the former AA_1, and the latter AA_2, and speak in the plural of Frege's Applicability *Arguments*.

Our focus will be on AA2: what would it prove? It would prove that (semantic) applicability requires of arithmetical statements that they: (i) express thoughts; and (ii) express true thoughts.

Strictly speaking, in neither AA1 nor AA2 do we find the claim that arithmetic is indispensable to empirical sciences, nor the claim that applicability requires the existence of mathematical objects such as the finite cardinals. It seems, therefore, that both arguments lack two of the features of CIA: INDISPENSABILITY-BASED and PLATONIST. We can then safely agree with Garavaso that AA2 is not, *per se*, an indispensability argument (nor that AA1 is).

Still, AA2 supports stronger conclusions than the mere advice that applicability should be given some explanation in a proper account of arithmetic (as Garavaso acknowledges in passing, see the quotation in the introductory section). At the very least, it supports semantic realism (as it has been spelled out above) with respect to arithmetic, since its conclusion is that arithmetical statements are true, *i.e.*, express true thoughts. Despite not being a version of IA by itself, therefore, Frege's AA2 may have some more significant connection with the way indispensabilists argue for semantic realism and/or platonism for mathematics. In order to see this, we do not need to extract any other version of IA from Frege's other works. Rather, our task should now be to investigate whether there are versions of IA such that Frege (or a neo-Fregean) could well accept their premises if presented with them, *also* on the basis of what he says in *Grundgesetze* II, §91.

3. IA IN A FREGEAN FRAMEWORK: THE COMPATIBILIST HERESY

Let us call a "compatibilist" someone who dares to claim that a neat intersection can be found between the rationalist position heralded by Frege and the (non-nominalist) empiricist view championed by Quine, in so far as Frege (or a neo-Fregean) may well accept some form of IA as a viable way of arguing for platonism: in other words, that the kind of considerations that usually lead indispensabilists to argue for platonism (or, if not all, at least a subset of the considerations that is nonetheless able to deliver a platonist conclusion) are at least *compatible* with a Fregean framework. Let us see how far this apparent heresy can be given support.

3.1. The Holist/Naturalist IA

There is clearly no hint, in Frege, of anything like a holist conception of confirmation, nor, more importantly, of anything close to naturalism (at least insofar as this is characterized, as above, as including a principled distrust of *a priori* arguments in philosophy). Nothing is more alien to Frege's views than the idea that empirical sciences are our privileged, let alone our only, source of knowledge. This much being granted, it immediately follows that, long before any evaluation of the similarity between IA and AA2, no argument assuming NATURALISM among its (explicit or background) premises will be acceptable to Frege, or, for what matters, to a Fregean.[11] Indeed, it is a consequence of

[11] *Cf.* fn. 8. Confirmational holism seems also incompatible with Frege's views, at least insofar as it is taken to imply the in-principle revisability of any inner part of a theory, logical and mathematical parts included.

versions of IA which assume NATURALISM that it is a necessary condition for the existence of a mathematical object that it is the object of a (true and) applied mathematical theory. Now, as will appear more clearly later when we will discuss the so-called *Frege's Constraint*, Frege would easily accept that — at least as far as arithmetic is concerned — the principle regulating the applicability of a mathematical theory should be built into the very definitions of the basic notions of that theory. And he would then accept as a consequence that, given the extreme generality and utility of arithmetic, its objects would certainly enjoy the property of being the objects of a mathematical theory which is applied in the sciences. From this, however, it follows neither (i) that arithmetic should necessarily be applied, nor, more importantly, (ii) that the actual (contrast: possible) particular applications of arithmetic should be considered a necessary condition for the existence of its objects. It is one thing to claim that Frege bestowed an important role on the applicability of arithmetic (and mathematics in general). It is a very different thing to claim that current applications are to feature necessarily among our reasons for believing in the existence of its objects. Current applications (contrast: applicability) do not play any proper justificatory role in Frege's derivations of Peano Axioms in *Grundlagen* and *Grundgesetze*, nor should they be counted among the reasons Frege offers for identifying numbers with self-subsistent logical objects. The difference in focus between applicability and actual applications will prove essential to marking the difference between Fregean and empiricist arguments for platonism.

3.2. Non-naturalist IA

In recent times, versions of IA that do not appeal to naturalism have begun to appear.[12] I will base the following discussion on the argument presented and discussed in [Panza and Sereni, 2013],[13] which I shall here call the *Minimal Indispensability Argument* (MIA):

i. We are justified in believing some scientific theories to be true / We are justfied in believing that S is true;
ii. Among these theories, some are such that some mathematical theories are indispensable to them / M is indispensable to S;
iii. We are justified in believing that the former are true only if we are justified in believing that the latter are themselves true / We are justified in believing that S is true only if we are justified in believing M is true;

(MIA_r) ____________________________

iv. Therefore, we are justified in believing true the mathematical theories indispensable to these scientific theories / We are justified in believing M true.

[12] Oddly enough, despite the fact that IA stems from a Quinean framework, where naturalism is an everywhere-ongoing assumption, no such naturalist assumption is present in Putnam's [1971] formulation (*cf.* fn. 2). This is stressed forcefully in [Putnam, 2012].

[13] For another non-naturalist version of IA, see [Azzouni, 2009].

v. We are justified in believing that a mathematical theory is true only if we are justified in believing that its objects exist / We are justified in believing that M is true only of we are justified in believing that the objects of M exist;

(MIA_p) ___________________________

vi. Therefore, we are justified in believing the objects of those indispensable mathematical theories exist / We are justified in believing that the objects of M exist.

In this argument[14] no appeal, either implicit or explicit, is made to naturalism (or holism, for that matter). The argument only gives *sufficient*, but not necessary, conditions for either the truth of mathematical theories (conclusion (iv)) or the existence of their objects (conclusion (vi)). If we accept MIA as a plausible version of IA, the first obstacle for the compatibilist is removed: naturalism and holism will not prevent a Fregean from accepting any version of IA.

3.3. Defeasible Justification *vs* Foundations

This may be seen as too quick a conclusion, however, for two reasons.

First, even though MIA requires no appeal to naturalism, it requires appeal to some form of scientific realism in order to justify premise (i).[15] This will not pose any serious obstacle to the compatibilist, however. For it seems totally safe to credit Frege with what we would qualify today as a form of scientific realism, *i.e.*, with a conception of science as aiming at a true description of the world (both concrete and abstract). More generally, nothing prevents a non-naturalist Fregean from endorsing a realist attitude towards scientific theories.

But a much more serious objection for the compatibilist is looming large here. Let us grant for the sake of the argument that nothing in MIA is strictly unacceptable to a Fregean. Still, MIA delivers only *a posteriori* and *defeasible* justification in support of its conclusion(s), and this is by no means the sort of justification that a (neo-)Fregean would require for foundational purposes. Since its conclusions — either that mathematical theories are true and/or that mathematical objects exist — are central to a proper epistemological foundation for mathematics (hence arithmetic), it follows that

[14] MIA ties together two sub-arguments, MIA_r and MIA_p, and the reasons for splitting it into two parts will be made clear later. Notice also that MIA can have an alternative (non-epistemic) formulation, that can be obtained by deleting all occurrences of 'we are justified in believing' and adjusting the grammar in the indicative accordingly. Since the distinction between epistemic and non-epistemic versions of MIA would not add much in this context, I will leave it out of the present discussion. See [Panza and Sereni, 2013, Chapter 6] for more details.

[15] This may even be a very weak form of scientific realism, for (i) does not require any scientific theory to be true. However, it requires at least that we can be justified in believing some scientific theory to be true, and this is something that many anti-realists would deny. In the alternative (non-epistemic) formulation of MIA (*cf.* fn. 14), where the first premise would be 'There are true scientific theories / S is true', a stronger version of scientific realism would be required.

MIA — like other versions of IA — would be inadequate to a (neo-)Fregean as an argument for either semantic realism or platonism: and this is not because the argument is *unsound*, but because it is *too weak* for the purpose at hand.

This, however, should not stop the ecumenical inspiration of the compatibilist. For even if it is correct to distinguish between *defeasible justification* and *proper logical foundation*, the former may still find its role in a Fregean framework. Let us recall one famous early passage in *Grundlagen* [1884, §2]:

> Of course, numerical formulae like $7 + 5 = 12$ and laws like the Associative Law of Addition are so amply established by the countless applications made of them every day, that it may seem almost ridiculous to try to bring them into dispute by demanding a proof of them … but the aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependance of truths upon one another. After we have convinced ourselves that a boulder is unmovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?

In this passage, Frege is not really arguing that the sort of justification for the truth of arithmetical formulae that comes from their 'countless applications' is to be entirely disregarded. He is not even telling us that 'countless applications' cannot deliver good grounds for believing in the truth of arithmetical formulae. He is, more sublty, telling us that 'proof' of arithmetical formulae — *i.e.*, the sort of step-by-step reduction of arithmetic to basic logical laws and definitions that he will sketch in *Grundlagen* and show how to obtain in detail in *Grundgesetze*[16] — will be required if our aim is not *only* to ascertain the truth of those formulae, but *also* — and, Frege might add, more importantly — to ascertain the 'dependance of truths upon one another'. It is reduction that is at stake here, not merely truth. In Frege's metaphor, it is the boulder's support that we are set to investigate, not the boulder's immovability, namely the truth of arithmetical formulae, of which we are already sufficiently convinced by experience.[17]

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[16] As well as the more modest derivation of Peano Axioms from full second-order logic plus HP and appropriate definitions that is proved by Frege's Theorem.

[17] A similar acknowledgement of 'various degrees of certitude' in 'apprehending a scientific truth' can be found also in the Preface to [Frege, 1879], from which all quotatons in this footnote come. Here, despite stressing that 'the most reliable way of carrying out a proof, obviously, is to follow pure logic', Frege acknowledges that

> perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. Hence we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how we can finally provide it with the most secure foundation.

The stress is here more on the fact that a question concerning the first, non-foundational, way of establishing a truth 'may have to be answered differently for different persons', whereas 'the second is more definite, and the answer to it is connected with the inner nature of the proposition considered'. Nonetheless, Frege is not explicitly ruling out the first 'degrees of certitude' as inapt also for *non*-foundational purposes. Thanks to Øystein Linnebo for bringing this passage to my attention.

To be sure, the idea that 'countless applications' could be taken as (defeasible) evidence for the truth of arithmetical formulae by no means conflicts with Frege's objections to empiricism. Here it will be enough to notice that Frege's criticism of the theses that — just to follow *Grundlagen*'s section titles — 'the definitions of individual numbers assert observed facts', that 'arithmetical truths [are] laws of nature', *i.e.*, 'inductive truths', that 'the number is a property of the agglomeration of things', that number is 'something subjective' like an idea; all objections to these claims could be accepted while at the same time accepting that applications of arithmetic offer good though defeasible reasons for believing in its truth.[18] Moreover, Frege's claims such as that (*cf. Grundlagen*, §87)

> The laws of number ... are not really applicable to external things; they are not laws of nature. They are, however, applicable to judgments holding good of things in the external world: they are laws of the laws of nature. They assert not connexions between phenomena, but connexions between judgments; and among judgements are included the laws of nature.

should also be seen as compatible with the idea that applications can (defeasibly) justify us in believing the truth of arithmetic. True, Frege is here stressing that *actual* applications, by themselves, are not in any sense constitutive of the laws of arithmetic, nor of the essence of numbers. Still this leaves completely open the possibility of arguing that those applications provide reasons for taking arithmetical propositions to be true.[19]

This may be seen as a somewhat relaxed reading of Frege's views, but it can easily be conceded to the compatibilist.[20] Once it is accepted, the compatibilist can maintain that even though MIA will not deliver the sort of 'proof' that Frege's foundational project asks for, it will at least offer sufficient reasons (defeasible as they may be) for

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[18]This line of reasoning has striking similarities with Field's [1989, p. 8] claim that the application of mathematical theories supply us with 'initial plausibility' for platonism:

> ... there are considerations that appear to favour the platonist... . These considerations have to do with the fact that mathematics is not just an autonomous discipline. Rather, mathematics has many applications outside mathematics — in scientific explanation, in the description of our observations, in metalogic, and in many other areas. It is the fact that mathematics appears indispensable in applications ... that provides the main source of arguments for platonism.

We will come back again to analogies with Field's strategy.

[19]This should not be read as meaning that successful applications *confirm* mathematical theories together with the scientific theories in which they are applied. Confirmational holism is still out of the picture.

[20]Support for this reading can be found in [Gabriel, 1996], according to whom Frege distinguishes between proof (*Beweis*) and justification (*Rechtfertigung* or *Begründung*). As Gabriel suggests (pp. 342–345),

> Frege expressly acknowledges the possibility of non-proving justifications and assigns the task to epistemology ... The distinction between logic and epistemology is marked in the later works by the terminological difference between 'a reason for something's being true' and 'a reason for our taking something to be true' ... For Frege, logical justification coincides with deductive inference, understood as 'proving' truths from truths. When he calls the reasons in a non-logical justification 'reasons for taking something to be true', he states that here the judgment does not stand in a deductive relationship to the reason ... Frege recognizes non-logical reasons as reasons, and thereby acknowledges epistemology as an argumentative basic discipline which is to be distinguished from the psychology of knowledge.

the truth of mathematical theories (and, pending qualifications, for the existence of the corresponding mathematical objects). But indispensabilists, in general, aim at no more than this: foundations are not among their objectives. It is then no objection to argue that IA, in whatever version, will not provide us with foundations. It is enough that it provides us with good enough reasons for semantic realism and/or platonism. This it does (if it is assumed to be sound), and Frege, until evidence to the contrary is offered, could acknowledge that it does.[21]

Let us also notice in passing that the fact that some versions of IA, possibly including CIA, are somehow based on inductive reasoning, or on an instance of IBE — the feature called above ABDUCTIVE — need not threaten the compatibilist. First of all, if what has been said so far is correct, Frege may well accept that the recollection of defeasible evidence for the truth of a mathematical theory can be obtained through inductive appreciation of the countless successful applications of that theory in science and everyday reasoning. This will make room for an induction-based argument, but not necessarily for an IBE-based one. Second, it is by no means mandatory that all versions of IA be based on abductive reasoning. It is not even so clear that CIA itself is an IBE-based argument. True, both Colyvan and others — *e.g.*, [Field, 1989, pp. 14–20] — have suggested that the kind of reasoning which is encoded in IA runs parallel to the kind of reasoning encoded in IBE-based arguments for scientific realism about theoretical entities. Nevertheless, the notion of explanation is not, at least on the face of it, involved at all in CIA. So one should rather look at arguments explicitly mentioning explanation, like the one offered by Baker [2009]. Be that as it may, it is clear that there are versions of IA that do *not* rely on IBE, nor on abductive arguments in general. Putnam's original formulation of IA is a case in point (see fn. 2 above). And MIA is clearly not abductive in character.[22]

3.4. Applicability, Indispensability and Realism

What about the remaining premises of MIA? Nothing seems to bar Frege from accepting premise (ii). Garavaso [2005, p. 167] is correct in claiming that Frege's AA is not INDISPENSABILITY-BASED, contrary to CIA, for clearly applicability does not entail indispensability. But it would not be pushing Frege too far to imagine that he would have accepted that (platonistically construed) arithmetic is indispensable to our scientific theories (especially if arithmetical truths are logical truths). He would likely have seen this as too obvious a datum to be pointed out explicitly.[23]

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[21] A *foundation* of arithmetic on inductively based evidence for the truth of its propositions will of course be rejected by Frege; see his remarks on induction in *Grundlagen*, §§9–10 and 17.

[22] Or not necessarily so: the role played by explanation can nevertheless be recovered once the notion of indispensability is suitably specified; *cf.* [Panza and Sereni, 2013, §6.3]. For more on the distinction between abductive and non-abductive minimal versions of IA, *cf.* [Busch and Sereni, 2012]. Notice that the fact that MIA's premise (i), and thus scientific realism, may be justified on inductive or abductive grounds; neither (a) prevents a Fregean from accepting MIA or scientific realism, nor (b) makes MIA itself an inductive or abductive argument.

[23] Notice that Frege's AA is in a significant way stronger than any IA, for even if IA were proved unsound by showing mathematics to be dispensable from science, a form of AA could still stand on

The compatibilist is only left with premise (iii). But it is easy to see that Frege may well accept it, and that AA_2 will have a major role in his acceptance. Premise (iii) states that a necessary condition for being justified in believing true a scientific theory, to which a mathematical theory is (indispensably) applied, is that we are justified in taking the latter to be true.[24] The material conditional expressed in this premise will be true as long as there is any independent reason for holding the relevant mathematical theories true. Frege had such independent reasons (delivered by the reduction of arithmetic to logic), so that for him — or for a neo-Fregean — premise (iii) would turn out to be immediately true. But uninterestingly so, one may object: nothing in this way of defending premise (iii) grounds its acceptance on the relations between the particular mathematical and scientific theories being considered.

Rejecting premise (iii) is tantamount to claiming — against [Putnam, 1975, p. 74] — that it is possible to be realists about science and at the same time anti-realists about mathematics. This might have been suggested by contemporary fictionalists as well as by the formalists of Frege's time, whose conception of the applications of mathematics does not hinge on its being true.[25] But AA_2 is advanced by Frege exactly to rebut the formalists on this point. Defending AA_2 entails defending premise (iii). Even if we concede that AA_2 amounts to no indispensability argument, we must acknowledge it as a means, easily available to Frege, for arguing in favour of a crucial premise of MIA.

3.5. The Heresy From Realism to Platonism

A final issue is left to the compatibilist. Frege's AA_2 is an argument to the effect that arithmetic expresses true thoughts, but IA is an argument for platonism, not just for semantic realism. AA_2 should also support arithmetical (mathematical) platonism, if the compatibilist heresy is to be eventually vindicated. But this it seems unable to do.

The objection, however, unnecessarily assumes that IA needs generally be an argument for platonism. But MIA_r seems a perfectly adequate version of IA for semantic realism, from which a version of IA for platonism can easily be obtained by the addition of further premises. Moreover, several commentators (*e.g.*, Paseau [2007], Azzouni [2004], and Pincock [2004]) have suggested that common versions of IA are more plausibly conceived as arguments for semantic realism only.

The compatibilist could thus rest content with the claim that the sole MIA_r is compatible with a Fregean framework. But she could also stress that MIA_r stands to MIA_p

the mere fact that mathematics is applicable. True, any good argument for the dispensability of mathematics will likely be flanked by an account of how false mathematical theories are usefully applied (as Field does), and this would prove Frege's AA unsound. However, one needs to distinguish conceptually between applicability and indispensability: at least in principle, IA could be blocked and some form of AA could still be in place. Thanks to Matteo Plebani for stimulating these remarks.

[24]In the alternative (non-epistemic) formulation of MIA — *cf.* fn. 14 — this condition would be stronger, since premise (iii) would be 'The former are true only if the latter are themselves true / S is true only if M is true'.

[25]This may be questioned with respect to Hilbert, since applicability is likely to require consistency and this is for Hilbert 'the criterion of truth and existence' (see Hilbert's letter to Frege dated 1899.12.29 in [Frege, 1980, pp. 39–40]). But for the formalists Frege is considering in the relevant sections of *Grundgesetze*, mathematical statements are not even truth-apt.

exactly as AA_2 stands to other independent arguments for platonism in a Fregean framework. Indeed, the argument in *Grundgesetze*, II, §91, by itself does not seem to rely on the thesis that numbers are (logical) objects, nor on the thesis that the sense of a numeral requires the numeral to be a singular term possessing reference, contrary to what formalists suppose. Frege's AA_2 should better be seen as a self-standing argument to the effect that arithmetic is true, to be then supplemented by other arguments — like those in the preceding sections of *Grundgesetze* or in the *Grundlagen* — in order to conclude that numbers are (logical) objects. And this neatly parallels the situation with MIA, where MIA_r must be enriched with the independent considerations in premise (v) if one is aiming at platonism.

In both cases these independent considerations will concern criteria by which it is possible to establish which objects exist if given statements (or a collection of them), whose logical form has been suitably displayed and in which apparently referential singular terms occur, are true. In the (neo-)Fregean case, this will involve interpretations of Frege's Context Principle and appropriate criteria for singular-termhood. In the indispensabilist's case, this is likely to involve the application of Quine's criterion for ontological commitment (*cf.* [Quine, 1948]) and reformulations in canonical notation. These differences aside, the dialectical situation is the same in both cases. Again, no serious obstacle seems to stand in the way of the compatibilist. Not yet, at least.

4. A COMPATIBILIST'S DILEMMA

The preceding discussion leaves us in a rather striking (and possibly worrying) situation. We started by acknowledging that some much-discussed versions of IA are immediately banned in a Fregean framework, but we seem now to have plenty of evidence that other versions may be perfectly acceptable to Frege or to a neo-Fregean. From this, one may wish to draw the following two conclusions. Firstly, that indispensabilists are fully justified when they look back at Frege as first suggesting the basic thought underlying IA. Secondly, and more boldly, that there may actually be a way of reconciling the rationalist and empiricist versions of platonism that are at stake. Both conclusions would be too hasty, though.

As concerns the first claim, we have stressed (agreeing in this with Garavaso) that the indispensabilists outright appeal to *Grundgesetze* II, §91, as anticipating any sort of IA proper is misplaced. But we also acknowledged that the possibility of retrieving from Frege's views some basic thoughts that, with suitable provisos, may lead to support to a form of IA is less dim than it appeared at first.

Notice that this is not just the clever trick of devising an argument so weak that even a Fregean could accept it. Rather, the idea is that if MIA, or any other non-naturalist cognate version of IA can qualify as a version of the indispensability argument reaching the very same conclusions by weaker assumptions, then the basic idea underlying indispensability arguments seems to be shareable, and shared, from fairly different perspectives. This basic idea is that the applications of mathematical theories to science can be explained only on the assumption that those theories are bodies of *true* statements, and (optionally) that they are about a domain of mathematical *objects*.

Vice versa, rejecting IA consists (or can consist), borrowing Dummett's words in a comment on Field's nominalistic program, in the attempt 'of explaining away reference to specifically mathematical objects' [Dummett, 1993, p. 435]. More generally, Dummett claims that once nominalist prejudices are left out of the picture, there is a significant consonance between Frege's and Field's projects:

> If we reject Field's all-encompassing nominalism, his programme takes on a different aspect. Much of the criticism directed at it falls away, once the task is no longer that of avoiding reference to all abstract objects; it continues to be of interest because it focuses on the problem which defeated both Frege and Russell, of either justifying or explaining away reference to specifically mathematical objects, and remains a problem even after the general objective of eliminating all reference to abstract objects has been discarded.[26]

Following these suggestions, one may easily attribute to Frege, if not a version of IA, at least an attitude towards mathematics for which the fact that reference to mathematical objects is unavoidable in applications (and, to be sure, in pure mathematics) plays a crucial role.

Still, despite the evidence gathered so far and the family resemblance between IA and AA2, one should not push the similarity between Frege's ideas on applicability and the basic idea underlying indispensability arguments too far. In the course of our discussion we have — on purpose — neglected the difference between *actual applications* and *applicability* of a mathematical theory. But this distinction proves to be crucial here.

It is now widely acknowledged that applicability plays a fundamental role in Frege's account of arithmetic; that, in other words, a properly Fregean account of arithmetic would abide by what Wright [2000, p. 324] calls 'Frege's Constraint':

> that a satisfactory foundation for a mathematical theory must somehow build its applications, actual and potential, into its core — into the content it ascribes to the statements of the theory — rather than merely 'patch them on from the outside'.[27]

Quoting from [Dummett, 1991, p. 60], Garavaso [2005, p. 169] rightly notices that Frege seems to adopt a double standard when he considers applications. On the one hand, he demotes applications when criticizing Mill's empiricist conception of mathematics in *Grundlagen*, whereas he praises the role of applicability when criticizing the formalists in *Grundgesetze*.

Now, the compatibilist heretical claim, as we have suggested it here, is that in an empiricist framework there are significant arguments for platonism and

[26]Interestingly enough, Dummett offers this shortly after the remark reported above in fn. 6.

[27]The last bit of this quotation is a quasi-quotation from Frege's *Grundgesetze*, §159, in which Frege is considering how real numbers should be defined. As Wright acknowledges, Dummett himself often stressed the relevance of this principle, for instance in [Dummett, 1991, pp. 272–273]. Whether it is advisable to abide by Frege's Constraint, either in arithmetic or in other mathematical domains, is a question over which we do not need to pause here. We will just assume that a fully Fregean account of a mathematical theory would abide by that constraint.

semantic realism about mathematics that are available to a Fregean (other points of disagreement notwithstanding), contrary to common opinion. What we have said so far on the compatibilist's behalf seems to go in her direction. However, it seems that a precondition for the acceptance of the compatibilist's claim is that the conception of mathematics that ensues from the conclusions of the relevant arguments will have to be immune to the criticism that Frege gives to the Millian empiricist in the *Grundlagen*. Here the distinction between applications and applicability becomes of the utmost importance. For we need to distinguish between two thoughts:

(1) that the determination of the senses of statements and the nature of objects, and thus of the truth of the former and the existence of the latter, is tied to *particular* applications of a mathematical theory; and
(2) that the definition of the fundamental concepts of a mathematical theory is such that it embodies the principle explaining the *actual and possible* applications of the theory.

Point (1) is what Frege rejects in objecting to Mill. He complains (*Grundlagen*, §9) that 'Mill always confuses the applications that can be made of an arithmetical proposition, which often are physical and do presuppose observed facts, with the pure mathematical proposition itself.'[28] If one indulges in this confusion between content and application, one is likely to end up conceiving the laws of arithmetic as inductive laws of nature, the 'symbol + in such a way that it will serve to express the relation between the parts of a physical body or of a heap and the whole body or heap' (*ibid.*), up until the absurd (in Frege's eyes) claim that 'the way in which number originates in us may prove the key to its essential nature' (§26). Thus if the conception of mathematics that is supported by IA hinges on something like (1), it will be outright foreclosed to Frege.

Compatibilists can go two different ways. On the one hand, they can revert to something like (2), and tie their consideration of applications to something as principled as Frege's Constraint. However, supporters of IA do not seem to adopt anything in the vicinity of Frege's Constraint. It is not even clear that it is at all coherent for an indispensabilist to adopt some such constraint, since it is a fundamental aspect of indispensabilist empiricism that the truth of a mathematical theory — and the existence of its objects — is not to be made dependent on the details of the formulation of the theory itself — nor on the alleged nature of its objects — but only on the fact that it is indispensably applied in true or well-confirmed scientific theories. No principled constraint on mathematical theories seems required or even legitimate in order to prove their truth (and the existence of their objects) — even though it is open to

[28] Similar considerations apply, according to [Frege, 1893–1903, Vol. II, §137], to Helmholtz's version of empiricism:

> Helmholtz is out to found arithmetic empirically, whether it bends or breaks. Consequently he does not ask: How far can one get without using facts of experience? but he asks: How can I most rapidly introduce some experimental facts? All who have this desire succeed very easily by confusing the application of arithmetical theorems with the theorems themselves. As though the questions about the truth of a thought and its applicability were not quite different! I can very well recognize the truth of a proposition without knowing whether I will ever have a chance to make use of it.

indispensabilists to acknowledge that some mathematical theories are devised with an interest to possible applications.

Alternatively, compatibilist indispensabilists can claim to have arguments available that defuse Frege's objections to the Millian, showing that the kind of empiricist framework wherein IA is located is immune to the fatal objections Frege made to Mill's naïve empiricism. Colyvan [2007, p. 109] does, as a matter of fact, defend his conception in exactly these terms:

> ... Mill's somewhat naïve empiricism found itself on the receiving end of a stinging attack from Frege ... Frege had many complaints but the most significant was that Mill had confused applications of arithmetic with arithmetic itself. ... Quine is not vulnerable to Frege's attack on Mill because Quine is not confusing mathematics with its applications. Rather, Quine is invoking the applications as a reason for taking the mathematics to be true.[29]

And Frege, the compatibilist may add, is in the same way 'invoking the applications as a reason for taking the mathematics to be true'.

Quine has of course a much subtler conception of mathematics than Mill's, and he would not indulge in the naïvety of thinking that the content (read: meaning) of mathematical statements is essentially determined by their actual applications. It is true, nonetheless, that on some occasions Quine rejected any ontological rights to the entities of unapplied mathematical theories, these latter been classified as mere 'mathematical recreation' [Quine, 1981, p. 400]. Quine [1990, pp. 94–95; 1995, pp. 55–56] later weakened his position on this issue, but he nonetheless suggested that the only reason for either reading mathematical statements as meaningful or for attributing to them a truth-value (indifferently either true or false) was just that of avoiding the unnecessary gerrymandering of our grammar that would violate the uniformity of our grammatical treatment of both applied and unapplied theories. Notice that even though indispensability is commonly offered as evidence for truth or existence, Quine is in these passages also speaking of *meaning*. If these passages of Quine's are to be taken seriously, then something more should be said to convince us that in a Quinean framework Frege's criticisms of Mill would not apply — to convince us, that is, that there is actually something important in common between Frege's and Quine's way of 'invoking' applications.

But something more seems to loom here. In fact, there are two aspects that should be taken into account.

In the first place, if Quine can avoid Mill's naïvety, it is not merely because he has some more elaborate way of arguing for the *truth* of (applied) mathematics. It is mainly because he has an extremely sophisticated machinery for approaching the problem of the *meaning* of mathematical statements — a machinery that was not available to Mill. Here is where the real difference between Quine and Frege lies. For while Frege would take applications to be (defeasible) evidence for the truth of statements, the

[29] Colyvan appends the following footnote here: 'Interestingly, this view can be traced back to Frege [1893–1903, §91]'.

determination of whose meaning is in principle wholly independent of any particular application, Quine would invoke applications as being evidence (defeasible as all the evidence a naturalist could hope for) for the truth of statements, whose meaning is determined in such a way that the role of a statement in past and current applications, and its position within the rest of the language, *do* enter significantly in the determination of its meaning.

In the second place, there is implicitly a striking difference between Frege's and Quine's invocations. Quine's invoking of applications as evidence for the truth of a mathematical theory is not merely the appeal to one among all available kinds of evidence: it is the invocation of the *sole* available evidence we may ever come to possess on this score. In Quine's invocation, naturalism is implicitly assumed, as suggesting or dictating where alone we should look for truth.

There is nothing new here. As is well known, both these two aspects are distant from Frege's views. It then seems that if applications can be evidence for truth at all in a Quinean framework, without leading by this to a Millian confusion between application and (determination of) content, it is only because holism (both semantic and confirmational) and naturalism are presupposed as background assumptions. But these background assumptions will immediately lead to arguments like CIA, which a Fregean will promptly reject.

The compatibilist seems thus caught in the following dilemma. Either she adopts a version of IA which assumes naturalism (and possibly confirmational holism), and then she supports a form of empiricism immune to Frege's objections to Mill, but she has to abandon the idea that the conclusions she reaches through IA will be available, *by the same route*, to a Fregean as well; or she adopts a version of IA along the lines of MIA, and then she can claim that her own path to semantic realism and platonism is compatible with Fregean views, but she still lacks an explanation of what makes her empiricism immune to Frege's objections to Mill.

5. CONCLUSIONS

If the preceding considerations are correct, indispensabilists appear to have, when they trace back some of their basic intuitions to Frege, much more room for maneuver than Garavaso concedes. However, this should not be taken as implying that indispensabilists are justified in attributing any version of IA to Frege on the basis of Frege's texts, as many authors suggest with reference to the argument in *Grundgesetze* II, §91. Rather, indispensabilists can show that Frege, or someone working in a Fregean framework, may well accept suitable versions of IA not only as sound, but also as pointing to a way of gathering defeasible evidence for the truth of mathematical theories and (possibly) the existence of their objects.

In doing this, compatibilists are going against the received opinion that IA, to quote Shapiro [2005, pp. 14–15], is 'anathema' for those who believe in the *a priori* character of mathematical truths. What they are pointing at is both that the real anathema is some common assumptions in IA — most notably, NATURALISM — and not the very basic intuition underlying the argument; and that both the indispensabilist and the Fregean programs share, in Dummett's words, the goal of 'either justifying or explaining away reference to specifically mathematical objects'.

This should make us aware that more consonance can be found on this issue in the two traditions championed by Frege and Quine. Nevertheless, compatibilists will face the dilemma that has been sketched above. A precondition for the Fregean acceptance of some version of IA is that the conception of mathematics it underlies is freed of some common objections. But in order to answer these objections, indispensabilists seem bound to revert to assumptions that a Fregean would immediately reject.

We should then be aware that the divide between the two camps may be narrower than we used to think. But we have not been shown yet that there is a way of filling the gap, however narrow it now seems.

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