IUSS
Scuola Universitaria Superiore Pavia

University School for Advanced Studies
IUSS

Doctoral Programme in Cognitive Neuroscience and Philosophy of Mind
XXXIII cycle

Weak Structuralism, Mutual Grounding and Quasi-Thin Objects: Steps Towards a New Taxonomy of Reality

Department of Humanities and Life Sciences

Ph.D student: Silvia Bianchi

Supervisor: Andrea Sereni

Academic Year: 2020-2021

CONTENTS

Acknowledgments

I thank the University School for Advanced Studies (IUSS) for funding my research-project with a Doctoral Scholarship during the years 2017-2020.

I am sincerely grateful to Andrea Sereni, who supervised my project, for his constant encouragement from a very early stage. His guidance helped me in all the time of research and writing of this thesis.

I wish to thank the reviewers of this work for their helpful comments, remarks and advices.

Special thanks go to Georg Schiemer, John Wigglesworth and the Vienna's research group of the ERC-Project *The Roots of Mathematical Structuralism* for the fruitful collaboration, of which this work has much benefited. My sincere thanks also go to my friend and colleague Joaquim Giannotti, whose insightful ideas and comments were an invaluable source of inspiration.

I thank my colleagues Luca Zanetti, for his precious advices, Maria Paola Sforza-Fogliani, Giacomo Zanotti, Elisabetta Tonini and Marco Facchin for the stimulating discussion throughout these years and for providing a pleasant and enriching work environment.

Last but not least, I am deeply grateful to my parents and to my life companion for their unconditional support and love in this intense journey.

Introduction: Prospects for Scientific and Mathematical Structuralism

Structuralism is a prominent approach in both the philosophy of science and the philosophy of mathematics. In the philosophy of science, structuralism is considered as the most defensible form of scientific realism, which combines the realist *no miracles argument* (Putnam, 1975) with the anti-realist *pessimistic meta-induction* (Laudan, 1981) with the aim of introducing the *best of both worlds* – where structures are identified with those parts of theories we are justified to be realist about. Metaphysically speaking, scientific structuralism offers a promising interpretation of contemporary physics, replacing the standard object-oriented metaphysics with an entirely structural ontology, which aims at providing a more suitable description of the recent developments of Quantum Mechanics. In the philosophy of mathematics, the structuralist theme has resulted from some significant changes within mathematics in the 19th century, supporting the view that mathematics primarily investigates structures and the relations between objects – not their internal nature, which is *abstracted away*. In both scientific and mathematical structuralism, there is a variety of competing views. A major distinction concerns *eliminative* and *non-eliminative* views. However, it is worth noting that while in scientific structuralism this distinction concerns a structuralism *without objects* versus a structuralism *with objects*, the same distinction has a different interpretation in mathematical structuralism, where it separates a structuralism *without structures* from a structuralism *with structures*.

The main purpose of this work is to introduce Weak Structuralism (WS) as a novel position in the structuralist framework. WS is intended to avoid some objections affecting Ontic Structural Realism (OSR) in the philosophy of science and Stewart Shapiro's *ante rem* structuralism in the philosophy of mathematics. Despite referring to different debates, these views have a number of similarities concerning the interpretation of structures, the nature of objects and the relationship between them. First of all, they are committed to a *robust* notion of structure: structures are understood as 'free-standing', self-subsistent entities made up of the relations between elements. This conception should be distinguished from a *formal* interpretation of structures as the abstract, mathematical components of theories, independent of their ontological content – which can be traced back to the historical foundations of both scientific and mathematical structuralism. Of course, the specific interpretation of structures varies in the two debates: in OSR, structures are *physical entities.* Unfortunately, there is no a canonical definition of what structures are in scientific structuralism but, standardly, structures have been identified with *web of relations*, *nomological structures* and *causal structures*. In the discussion of OSR, I simply refer to structures as the quantum entanglement structures in which quantum particles – when entangled – are in and as symmetry groups of group theory. In *ante rem* structuralism, structures are *sui generis* entities corresponding to *abstract patterns,* which are conceived as universals that can be instantiated by different systems. However, what ties OSR and *ante rem* structuralism together is the idea that structures are *fundamental* and *prior to* objects. The priority of structures over objects has led in both contexts to a sort of metaphysical decomposition of objects in structural terms. In OSR, physical objects are generally understood as the bearers of determinate relations, and they are either derivative or reducible to the relevant structures. Quantum particles are the most standard example. In a similar vein, Shapiro understands mathematical objects (i.e. natural numbers) as empty places or mere positions, whose identity is entirely determined by the structure they belong to.

Therefore, the two accounts appear significantly comparable for they entail that all that matters about physical and mathematical objects are their *structural properties*. I propose to spell out structural properties in terms of an *invariance* account, which has been advanced in both scientific and mathematical structuralism. In scientific structuralism, structural properties are the properties that remain invariant under symmetry groups transformations. In mathematical structuralism, structural properties can be identified with the properties which are inferred by a process of abstraction or, equivalently, as the properties shared by every (isomorphic) system instantiating the structure.

Still, an entirely structural interpretation of objects is seriously controversial and has raised similar criticisms in both views: OSR is subject to 'the relation without *relata*' objection, dealing with the nature of quantum particles in quantum entanglement structures and questioning how we can have a structure without the individuals making up this structure. Mathematical *ante rem* structuralism meets with the identity problem concerning the individuation of numbers in abstract structures: this problem specifically emerges in structures with non-trivial automorphisms, composed by distinct mathematical objects that turn out to be structurally indiscernible.

My objective is to show that both problems can be avoided insofar as: i) the metaphysical commitments of these views are more sharply defined ii) the structural and non-structural properties of objects are more precisely investigated. The emerging view is a form of Weak Structuralism, in which the fundamentality and the priority of structures are re-conceptualized in terms of *Mutual Grounding*, involving two distinct grounding claims: *Object Identity* (objects are grounded in structures for their *identity*) and *Structure Existence* (structures are grounded in objects for their *existence*). On the one hand, this is clearly a non-foundationalist relation – objects and structures are actually on a par. On the other hand, this relation resists some circularity problems concerning non-foundationalist views: after all, *Mutual Grounding* is not properly symmetric, for objects and structures are grounded in each other in two *distinct senses* – identity and existence.

By focusing on *Mutual Grounding*, it seems that a new category of objects can be individuated: I label such objects *quasi-thin objects*. As opposed to entirely structural objects, quasi-thin objects possess both structural properties and non-structural properties. While structural properties determine the essential identity of objects, i.e. what they really are as opposed to the other objects in the same structure, non-structural properties allow conceiving them as numerically distinguished *relata*, *conceptually* independent of the structure they belong to. I argue that *kind properties* offer a promising understanding of non-structural properties: in scientific structuralism they are state-independent properties such as mass or spin, distinguishing electrons from muons, etc. In mathematical structuralism they appear related to the use of numbers in *counting and measurement*, and they distinguish natural numbers from relative, rational, real numbers, etc. Interestingly, quasi-thin objects provide a possible strategy to avoid both the 'relation without *relata*' objection in OSR and the identity problem in *ante rem* structuralism. At the same time, the identity of physical/mathematical structures is settled independently of objects themselves, as stated by the third *Structure Identity* claim (structures are grounded for their identity in higher, more abstract structures) of *Mutual Grounding*, so that the priority of structures is preserved – consistently with a general structuralist perspective.

The structure of the thesis is as follows:

The first chapter deals with scientific structuralism; after providing an historical background, I distinguish Epistemic Structural Realism (ESR) and Ontic Structural Realism (OSR) and I focus on OSR. OSR comprises different approaches, which I define as *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR*. These views are subject to some general criticisms coming from the philosophy of science and to the more substantial 'relation without *relata*' objection, which affects specifically *Eliminative OSR* but easily generalizes to P*riority-based OSR* and some interpretations of *Moderate OSR*.

The second chapter examines mathematical structuralism, illustrating its historical foundations and some important taxonomic distinctions. Shapiro's *ante rem* structuralism will be discussed in detail, by assuming Shapiro's 'places-are-objects' perspective as a main theoretical premise. The second part of the chapter is devoted to the identity problem resulting from Shapiro's interpretation of objects and to some solutions that have been advanced, showing that they leave room for alternative strategies.

Chapter 3 provides a metaphysical toolkit to understand OSR and *ante rem* structuralism more precisely. First, I consider Metaphysical Foundationalism (MF) as the standard metaphysical explanation of reality. From this perspective, I present ontological dependence and grounding, favoring grounding in virtue of its stricter link with metaphysical explanation. Second, along the lines of Bliss and Priest (2018), I explore some promising non-foundationalist alternatives, such as Infinitism and Coherentism. Third, I outline Weak Structuralism (WS) as a further non-foundationalist option which can be applied in both the philosophy of science and the philosophy of mathematics.

With this groundwork, in chapter 4, I analyze some interpretations of OSR in terms of dependence, arguing that this notion *per se* provides compelling reasons to reject *Eliminative OSR*. I then recall *Priority-based OSR* and *Moderate OSR* which, though, are not without problems for they entail a very thin notion of objects and meet with some difficulties when it comes to individuating the relevant physical structures. For this reason, I refer to WS – expounded in chapter 3 – as a reasonable way of accounting for the relationship between objects and structures in scientific structuralism. Let us call it Weak Scientific Structuralism (WSS). In fact, WSS, along with the introduction of quasi-thin physical objects, seems to suggest a possible response to the 'relation without *relata*' objection and also a strategy to ground the identity of physical structures.

Chapter 5 returns to mathematical structuralism and Shapiro's *ante rem* approach: the interpretations of *ante rem* structuralism in terms of dependence (Linnebo, 2008) and grounding (Wigglesworth, 2018) give us useful tools to elucidate its metaphysical commitments. On this basis, I introduce WS in the mathematical framework, defining it Weak Mathematical Structuralism (WMS). WMS advocates an alternative solution to the identity problem (as applied to specific cases of structures with non-trivial automorphisms) and provides a more moderate understanding of *ante rem* structuralism, where more substantial quasi-thin mathematical objects are introduced and yet an *ante rem* individuation of structures is retained.

1. Scientific Structuralism: a Prelude

The present chapter illustrates Structural Realism in the philosophy of science, also defined as scientific structuralism. Scientific structuralism aims at providing the most suitable interpretation of our contemporary physics and comprises a broad family of approaches, which will be examined in the following sections. I will first provide a brief historical background in order to capture the main theoretical intuitions underlying scientific structuralism. As I am going to show, Epistemic Structural Realism (ESR) (see Worrall, 1989) naturally stems from these earlier structuralist views and offers a first significant interpretation of structuralism in more recent times, raising a revival of interest in the scientific structuralist debate. In the remainder of the chapter, I will focus on Ontic Structural Realism (OSR) – firstly articulated by Ladyman (1998) as opposed to ESR – and its main varieties, consisting of *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR*. In presenting these views, I will underline that they are related to different ways of interpreting objects and their relationship with the structure they belong to. The discussion will proceed by analyzing the main objections to OSR, focusing on its more substantial criticism, i.e. the 'relation without *relata* objection'. As I am going to show, even though this worry directly challenges *Eliminative OSR*, it seems to generalize also to *Priority-based* and *Moderate OSR* (French, 2010).

As the title of the chapter suggests, most of the ideas here presented are intended to pave the way to the investigation of mathematical structuralism – and specifically to Shapiro's (1997) *ante rem* structuralism, that has interesting analogies with OSR – which will be discussed in detail in chapter 2.

1.1. Scientific structuralism: the state of the art

The label 'structural realism' combines scientific realism – «the view that mature and genuinely successful scientific theories should be accepted as nearly true» (Psillos, 1999, p. xv) – with a structuralist approach, defined by an emphasis on «the structuralist features of scientific theories as a way of addressing epistemological and ontological problems in the philosophy of science» (Bokulich A., Bokulich, P., 2011, p. xi). In other words, even though scientific structuralism comes in a variety of flavors – captured by an extensive and intricate literature – what ties different approaches together is the central role attributed to the notion of *structure*. Of course, the project of clarifying the structuralist features of theories has been differently addressed.

A first important distinction concerns the goals of the structuralist inquiry: as cashed out by Psillos (2001, p. 13, my emphasis) «structural realism is meant to be a substantive philosophical position concerning *what there is in the world* and *what can be known of i*t». Originally, structural realism was introduced as a methodological and epistemic thesis (among others, see Poincaré, 1907, Cassirer, 1923, Russell, 1927, and, later, Worrall, 1989) claiming that structure is *all that can be known* about the unobservable physical world. Such approach was largely motivated by the attempt of preserving a realist attitude towards the process of theory-change, identifying a structural continuity in different subsequent theories. In the nineties, structuralism had a metaphysical turn (Ladyman, 1998), committing to the stronger claim that structure *is all there is* in the ontology. This position was articulated in response to an ontological problem, i.e. the fundamental underdetermination concerning quantum particles in Quantum Mechanics (QM) – consistent with two alternatives metaphysical packages: quantum particles as individuals and as not individuals. OSR's contributions are intended to break such underdetermination by introducing a third way, in which the concept of object itself is undermined and quantum particles are re-conceptualized in purely structural terms.

This major distinction entails a different interpretation of the notion of structure involved: on the one hand, the early structuralists refer to the *formal*, *abstract* and *logical-mathematical* aspects of scientific theories – to be contrasted with their ontological content. The focus on the formal features of theories remained crucial in Epistemic Structural Realism (ESR). On the other, later metaphysical interpretations of scientific structuralism (OSR and its variations) convey an *ontological* interpretation of structures as self-standing entities made up by the *relations* between elements – with obvious implications concerning the nature of objects belonging to such structures. The contrast between this two different interpretations has been nicely put by French (2006, p. 169) as follows:

> Broadly speaking, it [*structural realism*] consists of two fundamental strands: on the one hand, there is the identification of structural commonalities between theories; on the other, there is the metaphysical decomposition of objects in structural terms. Both have been pressed into service for the realist cause: the former has been identified primarily with Worrall's 'epistemic' structural realism; the latter with Ladyman's 'ontic' form. And both raise important issues of general interest within the philosophy of science and metaphysics, respectively. The former invites questions regarding the identification and appropriate representation of these commonalities; the latter touches on different views regarding the nature of objects, the constitutive role of properties and the seat of causal powers.

For now, let us start by investigating the historical foundations of structural realism and the main features of ESR.

1.1.1. Historical Background

Standardly, the first structuralist assumptions in the philosophy of science trace back to Poincarè (1905), Cassirer (1923) and Russell (1912;1927).

Poincarè (1905) has re-conceptualized the goals of science in a clear structuralist sense, suggesting that the 'the relations between things', as opposed to 'things between relations', are all that can be known about reality:

> Now, we daily see what science is doing for us. This could not be unless it taught us something about reality; the aim of science is not things themselves, as the dogmatists in their simplicity imagine, but the relations between things; outside those relations there is no reality knowable. (Poincaré [1905] 1952, p. xxiv, quoted in Bokulich, 2011, p. xi).

In a similar vein, Cassirer has emphasized a shift in the philosophy of science, from the ontological components of theories, to their structural and abstract import – denoting the relations between elements and the connections between our perceptual experiences which, in Cassirer's terms, are represented in intellectual *schemata*:

> This progressive transformation must appear unintelligible, if we place the goal of natural science in gaining the most perfect possible *copy* of outer reality. It is only owing to the fact that science abandons the attempt to give a direct, sensuous copy of reality, that science is able to represent this reality as a necessary connection of grounds and consequents. . . . Instead of imagining behind the world of perceptions a new existence built up out of the materials of sensation, it traces the universal intellectual schemata, in which the relations and connections of perceptions can be perfectly represented. Atom and ether, mass and force are nothing but examples of such schemata, and fulfill their purpose so much the better, the less they contain of direct perceptual content. (Cassirer [1923] 1953, pp. 164–165, quoted in Bokulich, *ibid*.).

It is though important to note that Poincaré's structuralism is generally associated with a form of conventionalism about science, and specifically about the geometry of space-time. According to Poincarè, geometry is a matter of *convention:* Euclidean and non-Euclidean geometries are inter-translatable, with the same things covered by different names.[1] A similar approach can be applied to physics: the shift between Fresnell and Maxwell's theories of optics is governed by conventional scientific principles (such as conservation of energy and least of action) which are accommodated by both theories and then accepted almost permanently, independently of empirical experiments. By contrast, Cassirer's structuralist claims are developed within an epistemic neo-Kantian perspective,[2] replacing the standard 'substantialistic' conception of reality – concerning substances, or physical objects, in the first place – with a functionalist view – in which functional relations only, encoded in the laws of nature, allow us to have an epistemic access to the basic physical entities. Despite specific differences, Massimi (2011, p. 2) observes that Poincaré's conventionalism and Cassirer's neo-Kantianism share a significant reconsideration of the notion of reference, which «is no longer identified with the unobservable entities that may (or may not) be the referents of theoretical terms, but

---

[1] See Poincaré (1908, p. 235:) «We know rectilinear triangles the sum of whose angles is equal to two right angles; but equally we know curvilinear triangles the sum of whose angles is less than two right angles. The existence of the one sort is no more doubtful than that of the other. To give the name of straights to the sides of the first is to adopt Euclidean geometry; to give the name of straights to the sides of the latter is to adopt the non-Euclidean geometry. So that to ask what geometry is proper to adopt is to ask, to what line is it proper to give the name straight?»

[2] In particular, Cassirer's intuitions rely on the Marburg School' interpretation of Kant.

with the mathematical structure of the theory»; more precisely, 'the burden of reference' is to be attributed to the conventional principles and functional laws which encode such mathematical structure in conventionalism and neo-Kantianism respectively.

However, the most substantial contribution to the structuralist debate is due to Russell (1912; 1927), who articulates structural realism as a full-fledged philosophical position. Russell's first structuralists assumptions, largely anticipating the epistemic relevance that structural realism assumes in ESR, can be found in *The Problems of Philosophy* (1912):

> Thus we find that, although the *relations* of physical objects have all sorts of knowable properties, derived from their correspondence with the relations of sense-data, the physical objects themselves remain unknown in their intrinsic nature, so far at least as can be discovered by means of the senses. (Russell [1912] 1959, p. 34).

Therefore, according to Russell, although *relations* between physical objects are knowable in virtue of their connection with the sense-data (i.e. the basic units of perception or *percepts*), we cannot know their intrinsic nature, but just the properties stemming from the relations between them.[3] This issue is further clarified in *The Analysis of Matter* (1927) which, along the lines of Poincaré and Cassirer, aims at providing a structural and epistemic analysis of relativity and quantum theory – in contrast with the standard object-oriented picture. In developing a causal theory of perception, Russell identifies the knowable properties of objects with their logical-mathematical properties:

> Thus it would seem that, wherever we infer from perceptions it is only structure that we can validly infer; and structure is what can be expressed by mathematical logic, which includes mathematics. (Russell 1927, p. 254, quoted in Bokulich, *ibid*.)

---

[3] The importance given to perception makes Russell's structuralist perspective an antecedent of what Psillos (2001) defines an *upward path* to ESR: assuming a 'bottom-up' approach, sensory experiences are considered as the primary source in the knowledge of the external world. On Russell's view, all that we can infer on the basis of sensory-experiences are structural features. Poincaré (1913) similarly starts from perceptual experiences to arrive at a structuralist knowledge of the world. The upward path to ESR is to be distinguished from a *downward path*, characterized by a 'top-down' approach: in this framework, scientific theories – and not perceptual experiences – work as a starting point to have an epistemic access to the structural, 'bottom' level of physical reality. This second conception is attributed to Poincaré (1905) and Worrall (1989). For a more comprehensive analysis see Psillos (2001), Votsis (2005) and Frigg and Votsis (2011).

The knowledge of the e*xternal world* is entirely structural. This claim relies on Russell's well-known distinction between *knowledge by acquaintance* and *knowledge by description*: while we know both qualities (properties and relations) and properties of qualities (structure) of *percepts*, we just know the structure of unperceived objects in the external world. This means that percepts are knowable by acquaintance, i.e. through a direct awareness: «we shall say that we have *acquaintance* with anything of which we are directly aware, without the intermediary of any process of inference or any knowledge of truths» (Russell 1912, p. 78). By contrast, objects in the external world can be known just by description, i.e. through inferences from our perceptions: «to know some thing or object by a definite description is to know that it is the so-and-so or that the so-and-so exists, i.e., that there is exactly one object that is so-and-so» (Russell, 1912, pp. 82–3). Significantly, all that can be inferred from perceptions is the *structure* of the world, which can be at best *isomorphic* to the structure of our perception.[4] As specified by Russell (1927, p. 250) «[t]he 'relation-number' of a relation is the same as its 'structure', and is defined as the class of relations similar [i.e. isomorphic] to the given relation».[5] The reference to class of relations makes the connection between *structural properties* and *mathematical properties* more explicit. The view according to which science tells about the structure of the world, and not about its nature, leads to an agnostic approach towards objects intended as external material objects: «[t]he only legitimate attitude about the physical world seems to be one of complete agnosticism as regards all but its mathematical properties» (Russell, 1927, p. 270). Therefore, to say that the intrinsic nature of objects remains unknown is not to say that objects lack any intrinsic properties – rather, what is suggested is the weaker claim that we cannot have an epistemic access to them.

---

[4] For a formal description of isomorphism see chapter 2, sec. 2.1.1.

[5] Redhead (2001) defines this notion of structure 'abstract', corresponding to an isomorphism class of structures which are similar/isomorphic to a given structure, and he contrasts it with a 'concrete' notion of structure, where structures are associated with a specific domain of objects and their relations.

1.1.2. Epistemic Structural Realism (ESR)

Epistemic Structural Realism as introduced by Worrall (1989) aims at combining the realist 'no-miracle argument' (Putnam, 1975) – according to which the success of science would be miraculous if not motivated by the approximate truth of scientific theories – with the anti-realist 'pessimistic meta-induction (Laudan, 1981) – pointing out that since past successful scientific theories turned out to be false, we have compelling reasons to expect that our current theories will themselves be abandoned. ESR is intended to break the impasse resulting from the attempt of accommodating both arguments:

> The main interest in the problem of scientific realism lies, I think, in the fact that these two persuasive arguments appear to pull in opposite directions: one seems to speak for realism and the other against it: yet a really satisfactory position would need to have both arguments on its side. The concern of the present paper is to investigate this tension between the two arguments and to *suggest* (no more) that an old and hitherto mostly neglected position may offer the best hope of reconciling the two. (Worrall, 1989, p. 101).

Worrall specifically refers to Poincaré's *Science and Hypothesis* (1905, pp. 160-162) which, in explaining the shift from Fresnel to Maxwell's theory of light, significantly anticipated the idea of a *structural continuity* between the two theories, accounted for by their differential equations – which capture adequately the *relations* between elements:

> This Fresnel's theory enables us to do today as well as it did before Maxwell's time. The differential equations are always true, they may be always integrated by the same methods, and the results of this integration still preserve their value [...] these equations express relations, and if the equations remain true, it is because the relations preserve their reality. They teach us now, as they did then, that there is such and such a relation between this thing and that; only the something which we then called *motion,* we now call *electric current*. But these are merely names of the images we substituted for the real objects which Nature will hide for ever from our eyes. The true relations between these real objects are the only reality we can attain.

In making this apparently 'forgotten thesis' (Worrall, 1989, p. 117) explicit, Worrall investigates more deeply the development of optics and what has got preserved in the fundamental shifts concerning the basic constitution of light: in the 18th century, light was conceived as constituted by material particles. This conception was replaced in the 19th century by Fresnel's theory of *luminiferous aether*: light was no longer understood as resulting from matter, but rather from vibrations carried out by a mechanical medium (the aether) pervading the space. Fresnel's theory itself was then rejected in favor of Maxwell's electromagnetic theory, claiming that light consists of waves propagating in the electromagnetic field.

As observed by Worrall, the standard picture of theory-change is one in which the empirical content of theories is retained, while substantial transformations occur at the theoretical level: in the present case-study, it seems correct to say that both Fresnel and Maxwell explained adequately optical empirical phenomena. In this picture, Maxwell's theory appears to be an 'extension with modifications' of Fresnel previous intuitions. However, theoretically, the two theories are clearly radically different, for they refer to mechanic vibrations and electric current respectively – with sharp changes in the theoretical apparatus describing them. It is exactly this view that largely inspired Laudan's (1981) pessimistic meta-induction, providing historical reasons to hold that present-day theories are likely to be replaced by different theories with a similar (though extended or slightly modified) empirical adequacy.[6] However, Worrall argues that such interpretation of the scientific continuity makes no concession to the no-miracles argument and then is far from providing an accurate description of science – which should be able to underwrite both the the pessimistic meta-induction and the no-miracles argument:

> How can there be good grounds for holding our present theories to be "approximately" or "essentially" true, and at the same time seemingly strong historical-inductive grounds for regarding those theories as (probably) ontologically false? (Worrall, 1989, p. 109).

Worrall claims that there is a third-way which allows us to have 'the best of both worlds' (p. 111), acknowledging the radical theoretical transformations in science without renouncing the approximate truth of scientific theories. That is because the pessimistic meta-induction relies on a false

[6] As observed by Worrall (p. 109), the theoretical core of the pessimistic meta-induction was already introduced by Poincaré (1905, p. 160): «the ephemeral nature of scientific theories takes by surprise the man of the world. Their brief period of prosperity ended, he sees them abandoned one after the other; he sees ruins piled upon ruins; he predicts that the theories in fashion today will in a short time succumb in their turn, and he concludes that they are absolutely in vain. This is what he calls the *bankruptcy of science*».

premise: in the basically 'cumulative' progress of science, there is something more than a preservation of the *empirical* content, but something less than a full continuity in the *theoretical* content – whereby the theoretical content is not reduced to the mathematical equations of a theory, but corresponds to the broader set of background theoretical assumptions which interpret the terms in the equations. It is not a matter of *content* at all. As already suggested by Poincaré, what Fresnel and Maxwell's theories have in common is the description of the *structure* or the *form* of light, despite being inconsistent with respect to its *nature*. After all, Fresnel was right not only about empirical phenomena, but also about the relations between them: for example, Fresnel correctly describes light as oscillations obtaining at certain angles and transmitted by a medium. Still, he misidentified deeply the nature of *what* oscillates and *what transmits* the oscillations: as showed by Maxwell, the vibrations are not mechanical, and there is nothing like an elastic, mechanic solid. In fact, any attempt to define the electromagnetic field in terms of aether was completely unsuccessful. By contrast, the oscillations are electric and magnetic strengths carried through a disembodied electromagnetic field. However, it is not surprising (i.e. it is not *miraculous*) that Fresnel theory was empirically predictive at the time: «the field in no clear sense approximates the ether, but disturbances in it do obey formally similar laws to those obeyed by elastic disturbances in a mechanical medium» (Worrall, 1989, p. 118).

Significantly, the Fresnel-Maxwell case-study easily generalizes and provides a common pattern reflecting theory-change. Actually, Worrall observes that the shift from Fresnel's to Maxwell's theory is unrepresentative in a sense, for it is an example in which the mathematical equations are *fully* preserved. The most common pattern is one in which the equations of a theory *re-appear* as limiting cases of the equations of the new theory. This suffices to re-establish the link between empirical success and truth, seriously undermined by the pessimistic meta-induction. In other words, structure becomes to be understood as the true – or the approximately true – part of scientific theories, i.e. the part we are justified to be realist about. On this basis, structural realism can be introduced as a substantial philosophical position: on the one hand, if we adopt structural realism, the 'pronounced death' of scientific realism seems to be a too drastic conclusion.[7] On the other hand, structural realism escapes some typical objections to scientific realism for structural realism does not engage at all in claims concerning the nature of objects.

Worrall's paradigmatic contribution leaves of course a number of questions open. First of all, what are exactly structures? Both Poincaré (1905) and Worrall (1989) suggest that structures are the

---

7 Worrall attributes this view to Fine (1984, p. 83): «Realism is dead...Its death was hastened by the debates over the interpretation of quantum theory where Bohr's non realist philosophy was seen to win out over Einstein's passionate realism».

mathematical equations of scientific theories, but a more specific definition remains to be accomplished. An option that is now quite standard among ESRists is that of identifying structures with Ramsey sentences, firstly introduced by Maxwell (1970) in order to resist the so-called 'Newman Problem' (Newman, 1928) affecting Russell's (1927) interpretation of structuralism.[8] Roughly speaking, Ramsey sentences allow for the elimination of non-observational terms in a theory by replacing them with existentially quantified predicate variables. Starting from the formalization of a theory in first order logic, $\prod(O_1,\ldots,O_n; T_1,\ldots,T_n)$, where *O*s terms correspond to observational terms and *Ts* terms to theoretical terms, the resulting Ramsey sentence is the following: $\exists t_1,\ldots, t_n\prod(O_1, \ldots,O_n ;t_1,\ldots, t_n)$.

I will not go in the details of this approach here [9] and I will focus on OSR as an alternative and more radical attempt of developing Worrall's (1989) assumptions in light of a more 'robust' notion of structure.

8 According to Newman (1928), Russell's claim that just the structure of the external world is knowable means that we know very little – if any – about the world. Newman's objection proceeds as follows: «any collection of things can be organised so as to have the structure W [where W is an arbitrary structure], provided there are the right number of them. Hence the doctrine that *only* structure is known involves the doctrine that *nothing* can be known that is not logically deducible from the mere fact of existence, except ('theoretically') the number of constituting objects» (Newman, 1928, p. 144, original emphasis). This makes it almost impossible to distinguish relevant structures from irrelevant ones. For this reason Massimi (2011, pp. 7-8) understands the Newman problem as a problem of *reference*: «Newman's problem is a problem about reference. Russell's structural realism is in the end a theory about how we can fix the reference of theoretical terms and be sure that they are genuinely referential, even if the objects at issue are unperceived and unperceivable. But, as Newman pointed out, Russell's structuralist solution was unable to single out reference, and hence unable to deliver on the original promise».

9 For a more comprehensive analysis of this approach, see Frigg and Votsis (2011). For some criticisms, see Demopoulous and Friedman (1985).

1.1.3. Ontic Structural Realism (OSR) and its main varieties

The question of which notion of structure is at play in structural realism as outlined by Worrall (1989) goes along with the broader issue of providing a more precise interpretation of this view as a whole. Worrall (1989) has clearly paved the way to the epistemic strand of structural realism. However, according to Ladyman (1998, p. 410), Worrall's paper is ambiguous, and thereby structural realism is not explicitly an *epistemic* position. In fact, at least some assumptions are consistent with a more substantial «metaphysical departure» (*ibid*) from scientific realism.[10] More importantly, if structural realism is really interpreted in epistemological terms – claiming that the knowledge of unobservable entities is purely structural – then it does not properly solve the problem of theory-change, which is a problem of *ontological* discontinuity; the no miracles argument remains to be accounted for. To this aim, structures must have «some grip on reality» (Ladyman, 1998, p. 419), or an *ontological* significance.

Moreover, scientific realism meets with another serious problem, which is actually overlooked by Worrall (1989), i.e. the *problem of metaphysical underdetermination*, in which metaphysics appears to be underdetermined by physics.

> Even if we are able to decide on a canonical formulation of our theory, there is the further problem of metaphysical underdetermination with respect to, for example, wether the entities postulated by a theory are individuals or not. There is, of course, much dispute about whether or not quantum particles, or spacetime points, are individuals. (Ladyman, 1998, p. 419).

This problem was firstly introduced by French (1989, 1998) with respect to Quantum Mechanics (QM). Quantum particles (such as bosons and fermions) are consistent with – at least – two metaphysical packages: on the one hand, they are individuals whose individuation relies on non-empirical facts, i.e. *primitive thisness* or *haecceity*. On the other hand, they are non-individuals, devoid of well defined identities. In order to understand this problem, some preliminary assumptions are needed. Objects can be individuated in accordance with the Leibnizean Principle of Identity of Indiscernibles (PII), which plays a crucial role in the discussion on both scientific and mathematical structuralism. PII is formally expressed as follows:

---

[10] Ladyman (1998, p. 410) specifically refers to the following passage: «on the structural realist view what Newton really discovered are the relationships between phenomena expressed in the mathematical equations of his theory» (Worrall, 1989, p. 122.)

$$\forall x \forall y\ [(\forall P)(Px \leftrightarrow Py) \rightarrow (x=y)]$$

More informally, the principle states that if *x* and *y* have the same properties, then they are the same object. A first issue concerns which properties are at hand. Three options are on the table: i) intrinsic properties; ii) intrinsic and relational properties; iii) intrinsic and relational properties, where relational properties are intended to include spatial properties. In both classical and quantum physics, particles of the same kind – say, for example, two electrons – possess the same intrinsic properties and stand also in the same relations, thus being indiscernible in the first two senses (i); (ii). However, in classical physics, particles are discernible by means of their spatio-temporal location, for their spatio-temporal trajectories cannot overlap and classical objects are seen as 'impenetrable'. Therefore the third (iii) interpretation of PII is not violated. The problem with QM is that quantum particles are indiscernible even with respect to their spatial properties, so that they do not obey PII in none of the three senses (i); (ii); (iii) specified above. This specifically stems from the so-called Indistinguishability Postulate (IP), related to the statistics of quantum physics, showing that a permutation of quantum particles does not give rise to a new arrangement.

(IP): «If a particle permutation *P* is applied to any state function for an assembly of particles, then there is no way of distinguishing the resulting permuted state function from the original unpermuted one by means of any observation at any time» (French, 2019, sec. 2).

As explained by French (2019, sec. 4) «no measurement whatsoever could in principle determine which one is which». Therefore, the antecedent of PII is true, and quantum particles – sharing their intrinsic, relational, and even spatial properties – turn out to be exactly similar. This situation clashes with QM, which actually refers to two-particles systems, and results in the aforementioned metaphysical underdetermination: either quantum particles are non-individuals or they are individuals whose individuality needs to be established by means of *primitive thisness* or *haecceity* – transcending the qualitative properties of particles.[11] As pointed out by Ladyman (1998) and French and Ladyman (2003), breaking such underdetermination requires a *metaphysical shift*, in which the concept of object itself is undermined and a question concerning individuality simply does not arise.
On this basis, Ladyman (1998) claims that structural realism, in order to solve both the problem of theory-change and the problem of metaphysical underdetermination, should be developed as a *me-*

[11] Primitive thisness or haecceity are in principle consistent with QM (see French and Redhead, 1988).

taphysical position, defined Ontic Structural Realism (OSR). This point is clarified further in French and Ladyman (2003a): while Worrall (1989) presupposes a dichotomy between *structure*/*form* on the one hand and *content*/*ontology* on the other hand, OSR should be advanced «as [...]a reconceptualisation of ontology, at the most basic metaphysical level, which effects a shift from objects to structures» (French and Ladyman, 2003a, p. 37).

On this view, structure is taken to be «primitive and ontologically subsistent» (Ladyman, 1998, p. 419). This interpretation of structure differs deeply from that presupposed by Worrall (1989), consisting in the formal mathematical equations of a theory. Broadly speaking, structures in OSR are webs of relations: «physical structures [...] capture the natural – that is, causal-nomological – relations among the objects of a system» (French, 2006, pp. 175-176), where these objects are at best understood as *points of intersection*. According to Ladyman (1998), French and Ladyman (2003) and Ladyman and Ross (2007), structures have an inherently *modal* character, describing the modal relations between (both actual and possible) phenomena.[12] In fact, structures, in order to be ontologically relevant, «must go beyond a correct description of the actual phenomena to the representation of the modal relations between them» (Ladyman, 1998, p. 418).[13]

Ladyman (1998) and French and Ladyman (2003) refer to Weyl's (1931) interpretation of QM and introduce group-theory as a promising mathematical characterization of physical structures in structural realism. Structures correspond to symmetry-groups of group-theory, which can be translated into one another. Taking a symmetry to be an invariant transformation of a structure, symmetry groups are defined as follows:

> A group of symmetry transformations is a mathematical object which consists of the set of transformations, including the identity transformation and the inverse of each transformation, and the operation of composing them, where the result of two composed transformations is itself in the original set. (Ladyman, 2020, sec. 4.1).

In this picture, an interpretation of *objecthood* is also provided: objects emerge as *invariants* in symmetry-groups transformations. On this view, the two metaphysical packages (individuals and non-individuals) underdetermining quantum particles are seen as two *representations* of the same structure, which is ontologically basic. In other words, the notion of object – and its two alternative

[12] This conception is to be contrasted with the standard *set-theoretic* notion of structure, where structure is composed by objects and their intrinsic properties, on which the structure supervenes.

[13] Such approach is directly based on Giere's (1985) constructive realism.

interpretations – is replaced by an entirely structural ontology. I will come back to the role of symmetry groups and mathematical representation of physical structures in chapter 4.

A metaphysical interpretation of Structural Realism – along with a more substantial conception of structures – has raised a wide debate, comprising a variety of OSR-views. These approaches have been captured by different taxonomies (Ainsworth, 2010; French, 2010; Esfeld and Lam, 2011). In what follows, I am going to focus on three main variations, even though there exist many other positions in between: 1) *Eliminative OSR*; 2) *Priority-based OSR*; 3) *Moderate OSR*. Positions (1)-(3) entail different conceptions of objects and their properties within the structures they belong to. Still, it is important to observe that each of them aims at contrasting the standard object-oriented metaphysics, according to which objects are equipped with intrinsic properties and physical relations supervene on objects – a view that is standardly defined *Humean supervenience* (Lewis, 1986). Intrinsic properties are the «properties that are independent of whether the object is alone or accompanied by other objects» (Esfeld and Lam 2011, p. 144). Supervenience, on the other hand, is a relation of necessary covariation: if A supervenes on B, then there cannot be an A-difference without a B-difference. (for a general overview, see McLaughlin and Bennett 2018). The idea that relations supervene on intrinsic properties of objects is deeply challenged by QM, for quantum particles in entanglement states (i.e. two electrons in a singlet-state) are entirely defined by their structural, state-dependent properties. Such claim has been clarified by Teller (1989) in terms of *non-supervenient relations*. The same point is stressed by Ladyman and Ross when they introduce the notion of modal structure: the world has «an objective modal structure that is ontologically fundamental, in the sense of not supervening on the intrinsic properties of a set of individuals» (2007, p. 130) Let us now consider *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR* in more detail:

1) *Eliminative OSR*:[14] this is the original interpretation of OSR (French and Ladyman, 2003; Ladyman and Ross, 2007; French, 2010). There are relations without objects or *relata* standing between them; in slogan form, «there are no things, and structure is all there is» (Ladyman and Ross 2007, p. 131). We *know* just structure because the world *is* just structure, and «there is nothing else to know» (Ainsworth, 2010, p. 50). In this way, *Eliminative OSR* aims at filling the gap between epistemology and metaphysics underlying ESR, in which objects – if they exist – remain inscrutable. Ladyman and Ross (2007) themselves define their approach 'eliminative': that is because objects «[...]have been purged of their intrinsic nature, identity and individuality,

[14] The label *Eliminative OSR* is from Psillos (2001).

and they are not metaphysically fundamental» (p. 132). They are at best theoretical constructs useful for constructing approximate representations of the world. On this view, the very notion of object is abandoned and structures are understood as objectless networks; therefore, there is no question concerning the existence of objects, for objects themselves are no longer part of the fundamental ontology of the world. Similarly, a discussion concerning the properties of objects simply does not arise in the eliminativist picture. Among OSR-views, *Eliminative OSR* is the most troublesome. In section 2.1.1. I will address the main objection to it, i.e. the 'relation without *relata*' objection.

2) *Priority-based OSR*:[15] This position is now the most standard in OSR debate and it is also considered a more plausible re-elaboration of *Eliminative OSR*. On its broadest construal, *Priority-based OSR* claims that there are both objects and relations, but relations are ontologically prior to objects; this means that relations bear the ontological weight, determining the existence of the *relata* composing them. Even if this view admits intrinsic properties of objects, they are not significant, because they do not allow distinguishing quantum particles. In fact, the identity of objects is also reconstructed from relations, consistently with the idea of a *contextual identity* for quantum particles, derivative on the relations in which they stand (Stachel 2002; Ladyman 2007) and sufficient to support a *thin* notion of objecthood. The idea of thin objects as mere *nodes* or *positions* has been originally elaborated by Saunders (2003) in terms of a weaker form of the Principle of Identity of Indiscernibles (PII) and a weak notion of discernibility for quantum particles. Such proposal is based on Quine's (1960, p. 230) distinction between different grades of discernibility: *absolute*, *relative* and *weak discernibility*. Two objects are absolutely discernible if there is a one-variable formula which is true of an object and not of another; relatively discernible if there is a two free-variables formula which applies to them just in one order; weakly discernible if there is a symmetrical but irreflexive relation holding between them. Consider two fermions in a singlet-state having an opposite spin: while the two particles cannot be either absolutely or relatively discernible (they are indistinguishable in isolation, since their permutation leaves the state they are in unchanged) they are weakly discernible in virtue of the two-place irreflexive relation holding between them (i.e. having opposite direction of each component of spin to...). The fact that two objects bear an irreflexive relation *a*R*b* shows that they are two and not just one, because nothing can stand in an irreflexive relation with itself. On these

[15] The label is from McKenzie (forthcoming); by contrast, French (2010) defines this position Weak Structural Realism.

grounds, weak PII introduces a further formulation of the principle, one in which symmetric but irreflexive relations (iv) are admitted in the set of the relevant properties of objects, in addition to intrinsic (i) relational (iii) and spatial properties (iii). However, while weak PII numerically distinguishes two fermions with an opposite spin, it cannot individuate them, i.e. it is still impossible to establish which particle is which.[16]

3) *Moderate OSR*: on this view, there are both objects and relations, without there being any ontological priority between them; objects and structures are both fundamental entities of reality. On the one hand, in contrast to *Eliminative OSR*, *Moderate OSR* admits objects in the ontology. On the other hand, objects are understood as the bearers of relations, so that a broader notion of objects is actually introduced. This position has been outlined by Esfeld and Lam (2008) as follows:

> Neither objects nor relations (structure) have an ontological priority with respect to the physical world: they are both on the same footing, belonging both to the ontological ground floor. (Esfeld and Lam 2008, p. 31).

This formulation captures the idea that objects and structures are on a par – objects and structures can neither *exist* nor be *conceived* without each other. In fact, objects are defined by the relations making up the structure. At the same time, structures exist in the *physical* world as relations between objects. *Moderate OSR* is also committed to the claim that the *identity* of objects is determined by the relations and vice-versa. In contrast to a standard metaphysical view, in which objects are given *at first*, and *then* they enter into the relevant relations, in *Moderate OSR* «we get the *relata* and the relations at once as the internal structure of a whole, neither of them being eliminable or reducible to the other one» (Esfeld and Lam, 2008, p. 34). Esfeld and Lam (2011, p. 146) mention two main implications of this account: first, on this view, the numerical diversity between objects – to be distinguished from their very identity, that is entirely relational – is taken as primitive, since it is neither grounded in intrinsic properties, nor in relations: if objects and

[16] Weak PII has been also extended to bosons (Muller and Saunders, 2008; Muller and Seevinck 2009), although some details are contentious (see Bigaj, 2015a and 2015b; Caulton, 2013; Huggett and Norton, 2014; Norton 2015).

relations are on the same ontological floor, then we need at least a very minimal notion of objects to start with. Second, *Moderate OSR* admits extrinsic, relational properties as part of the fundamental ontology: «if there are physical relations among objects as *relata*, these objects have relational properties, and the other way round» (*ibid.*).

This position has been applied to both quantum entanglement (Esfeld, 2004) and space-time (Esfeld and Lam, 2008).

1.2. Questioning OSR

Ontic Structural Realism has been subjected to a variety of objections.[17] Some of them are general, i.e. they affect OSR as broadly understood. Others are more specific, for they question particular forms of OSR. Let us start with the former, undermining OSR as a *metaphysical* position. A first standard objection, formulated by Psillos (2001), is that OSR is too *metaphysically visionary*: QM does not necessarily commit to abandon objects at all, for quantum particles in entanglement states can be individuated in virtue of their primitive thinness or haecceity. On these grounds, ESR seems a much more defensible position, positing structures as composed by (unknown) individuals. According to Psillos (2001, p. S23),

> One way to read SR (*ed. Structural Realism*) is to take it as a modest epistemic thesis that emerges from looking into the history of scientific growth. There is no heavy metaphysical machinery behind it, nor any absolute claims about what can or cannot be known. It is just a sober report of the fact that there has been a lot of structural continuity in theory-change.

Ladyman and Ross (2007) and Ladyman (2020), however, argue that it is far from clear that either standard realism or alternative approaches are more adequate than OSR; «a strong burden of proof is on those who would abandon traditional metaphysics» (Ladyman, 2020, sec. 5.) and a form of verificationism should be applied in metaphysics: according to Ladyman and Ross (2007), a verificationist approach suggests to reject the existence of objects we have no reason to believe – objects

---

[17] Other objections to Structural Realism seem to target ESR: that according to which it collapses in standard realism (Psillos, 1995) and the idea that structure is also lost in theory change (Stanford, 2003). Another objection is that Structural Realism is applicable just to physics, but Ladyman and Ross (2007) and French (2010) engage in extending this position to social sciences and biology respectively.

we cannot know, as argued by ESR. Adopting an agnostic attitude, in fact, would commit one «to be agnostic about a literal infinity of matters—whatever anyone can conceive without contradicting physics» (Ladyman and Ross, 2007, p. 131).[18]

Another challenge to OSR has been introduced by Chakravartty (2003): OSR, by eliminating – or significantly weakening – objects, cannot account for causation (i) and for the idea that some properties and relations clearly cohere (ii). Concerning i), the following worry emerges:

> One of the most important explanatory roles served by objects is to provide a means of change. Objects have properties, and it is because they have these properties that things happen to them [...] How does an objectless ontology account for change? (Chackravartty, 2003, p. 872).

French (2006) replies to (i) by addressing the notion of objective modal structure (see section 1.1.3):

> «Indeed, she can respond to Chakravartty's concerns by insisting that the explanatory buck stops at a point down the chain before we reach objects. That is, she can insist that this active principle lies with the relations and properties themselves, and it is these which carry the clout [...]. The OSR [...] would simply insist that rather than thinking of this description in terms of causally interacting physical objects, we give an appropriately structural description involving causal relationships. (French, 2006, p. 181).

A similar idea is advanced by Esfeld and Lam (2011, p. 156):

> One sense in which the structures can be modal is by being causal [...] Taking the structures to be causal is to say that insofar as there are concrete physical relations as the ways in which the fundamental physical objects exist, these are powers or dispositions to bring about certain physical effects.[19]

---

[18] In particular, Ladyman and Ross (2007) follow Peirce's interpretation of verification: «This verificationism consists in two claims. First, no hypothesis that the approximately consensual current scientific picture declares to be beyond our capacity to investigate should be taken seriously. Second, any metaphysical hypothesis that is to be taken seriously should have some identifiable bearing on the relationship between at least two relatively *specific* hypotheses that are either regarded as confirmed by institutionally *bona fide* current science or are regarded as motivated and in principle confirmable by such science». (Ladyman and Ross, 2007, p. 29).

[19] For a more extensive understanding of causation in OSR, see Ladyman and Ross (2007, chap. 5)

At the same time, doing without objects leaves unexplained why certain properties tend to cohere and form a 'unity' (ii) – it seems quite implausible that this is just a matter of coincidence. The best explanation is just that these properties and relations cohere because there are objects in the structural ontology. In this respect, French (2006) appeals to a principle of mere *compresence* – related to the idea that coincidences, and nothing more, happen in physics – which «ties together aspects of different structures» (p. 185), giving rise to the objects we define quarks, electrons, etc. as mere *nodes* in these structures.

Another serious objection to OSR claims that OSR makes physical structures collapse into mathematical structures, committing to a form of Pythagoreanism (Dipert, 1997) – according to which the structure of the world *is* mathematical. After all, the standard criteria for identifying physical/concrete structures – such as spatio-temporality and causality – are hardly applicable to the quantum structures OSR deals with, in which mathematical components (i.e. symmetry groups of group-theory) play a crucial role. The objection goes as follows: if the structuralist features of mathematical theories are essential to physical ontology, then physical and mathematical structures turn out to be identical. Differently put, their distinction cannot be established in purely structural terms (Van Frassen, 2006).[20] However, categories such as spatio-temporality and causality are largely inaccurate when it comes to interpreting modern physics (Ladyman and Ross, 2007, p. 160). For this reason, Ladyman and Ross (p. 158) refuse to engage in drawing the distinction between the mathematical and the physical: «The 'world-structure' just is and exists independently of us and we represent it mathematico-physically via our theories». By contrast, French and Ladyman (2003b) deny explicitly that OSR commits to the view that the world's structure is mathematical and argue that physical structures can be distinguished from the mathematical ones because just the former stand in a relationship of partial isomorphism with physical phenomena – in fact, in mathematics, such isomorphisms obtain between mathematical structures, but not between mathematical structures and the physical world (French and Ladyman, 2003b, p.75).[21] Moreover, as specified above,

---

[20] This results from Van Frassen's (2006) idea that OSR ends up with no structure at all: «once adopted, it is not to be called structuralism at all! For if there is no non-structure, there is no structure either. But for those who do not adopt the view, it remains startling: from an external or prior point of view, it seems to tell us that nature needs to be entirely re-conceived». (Van Frassen, 2006, pp. 292-293).

[21] However, Ainsworth (2010) has raised a serious objection to this way of drawing the distinction between physical and mathematical structures: after all, provided that there exist isomorphisms between physical structures and physical phenomena, physical phenomena are themselves structures; but then «there are, trivially, mathematical structures isomorphic to them» (Ainsworth, 2010, p. 51) So, it is not true that just physical structures stand in an isomorphism relation with physical phenomena.

physical (modal) structures are causal, although the sense of causality here at stake differs from the standard one – where objects, and not relations, have 'causal powers'.
Lastly, a specific objection to *Eliminative OSR,* i.e. the relation without *relata* objection*,* deserves a separate investigation in the next section. In fact, not only is this criticism the most serious challenge to OSR, but it also seems to have unpleasant consequences for other versions of OSR.

1.2.1. Questioning Eliminative OSR

Eliminative OSR, which holds that «structure is all there is» and admits no objects in the ontology, faces a serious objection concerning the coherence of having relations (structures) without *relata* (objects). This criticism has been raised by several authors (Cao, 2003; Dorato, 1999; Psillos 2001, 2006; Busch, 2003; Morganti 2004; Chakravartty, 1998, 2003). Esfeld and Lam (2011, p. 148) distinguish three main interpretations of the objection: i) a *metaphysical* one, concerning the very intelligibility of a metaphysical position which posits structures and yet rejects the *relata* making up these structures; ii) an *empirica*l one, claiming that the physical evidence in QM by no means motivates the elimination of objects; iii) a *logical* one, according to which first order logic requires objects to quantify over, in a set-theoretic spirit.

Let us start with the metaphysical interpretation (i). In Chakravartty's (1998, p. 399) words «one cannot intelligibly subscribe to the reality of relations unless one is also committed to the fact that some things are related». In a similar vein, Busch (2003, p. 214) maintains that «the very idea of structure presupposes some elements that go together to make up that structure. A relation might take anything as its *relata*, but it always takes something». Without such *relata*, OSR's conception of structure is somehow mysterious: OSR combines concrete elements – spatio-temporality and causality – with abstract ones – structures as ontologically subsistent and 'free-standing' – [22] thus leading to considerable difficulties in understanding what structures in OSR exactly amount to. Esfeld and Lam (2008, p.31) and Chacravartty (2003, p. 871) also argue that if OSR does without objects, then it is hardly intelligible as an interpretation of the *physical* world, for the notion of concrete structure itself needs *something* to be related.

[22] This introduces interesting analogies between OSR and Shapiro's (1997) mathematical *ante rem* structuralism (see chapter. 2, sec. 2.2.4).

The *empirical* implications of eliminating *relata* (ii) are underlined by Ainsworth (2010, p. 53): although quantum entanglement states provide a serious counter-example to the standard object-oriented in metaphysics, it is far from obvious that *Eliminative OS* can make sense of the physical features of QM. In fact, even the most deflationary interpretations of QM presuppose objects and their properties. Therefore, the available empirical evidence does not actually justify the elimination of objects from the ontology and – even worse – quantum particles do not really cohere with an entirely relational metaphysics which is object-free and property-free.[23] In particular, while some intepretations of QM do away with objects (Ghirardi, 2007) – at least as fundamental ontological units – the elimination of properties from quantum physics is far more controversial. In fact, Ainsworth (p. 54) points out that entangled quantum particles still have state-independent properties such as mass or charge and that there are cases in which quantum particles are not entangled and yet have state-dependent properties (for a more detailed argument, see Ainsworth, 2010, sec. 3.1). A related criticism to *Eliminative OSR*, concerning the difficulties of reducing state-independent properties to a structural, group-theoretic characterization (Wollf, 2011; MacKenzie, forthcoming) will be presented in chapter 4, sec., 4.2.1, when discussing the relationship between objects, structures and state-independent properties interpreted as kind properties, i.e. the properties distinguishing electrons, muons, etc.

The last aspect of the objection is the logical one (iii). Bain (2009) observes that the objections against the prospects of positing relations without *relata* rely on a set-theoretic conception of structures, in which structures are understood as sets of objects, and favors category theory as the appropriate framework to conceptualize *Eliminative OSR*. Still, Esfeld and Lam (2011, p. 148) put into question both that category theory is independent from set-theoretic concepts and that it is really relevant for QM.

Eliminativists endeavoured in making sense of the ‘relations without *relata*’ intuition by interpreting structures as universals (Stein 1989; Psillos 2006) or arguing that the *relata* of the relations turn out to be structures themselves, with relations all the way down (Ladyman and Ross 2007; Saunders 2003). Either way, such proposals are contentious, leaving room for the more defensible *Priority-based OSR* and *Moderate OSR*, which include objects in the structural ontology. However, as I am going to show in the next section, both accounts rest on a very *thin* conception of objects and then are subject to variations of the 'relation without *relata*' objection.

---

[23] The empirical aspect of the 'relation without *relata*' objection is clearly related to Psillos's (2001) objection according to which OSR is too metaphysically visionary (see section 1.2).

1.2.2. Related Objections to Priority-based OSR and Moderate OSR

As opposed to *Eliminative OSR*, *Priority-based* and *Moderate OSR* admit both objects and structures in the ontology. What varies in the two positions is the relationship between the two: while *Priority-based OSR* takes structures to be primary and objects to be secondary or derivative on such structures, in *Moderate OSR* objects and structures are on the same ontological floor. Despite these differences, both views reduce objects to the relations in which they stand: in fact, recall that in *Priority-based OSR* objects emerge from structures as mere nodes in the relations. Similarly, *Moderate OSR* understands objects as the bearers of relational properties – in contrast with the received view, in which objects are characterized by their intrinsic properties. Consider *Priority-based* OSR first. As pointed out by Esfeld and Lam (2011, p. 148), this approach is not immune from the three aspects (i-iii) of the 'relation without *relata*' objection:

> [...] the commitment to an ontological priority of relations over relata again invites the above mentioned objection in all its three aspects, for if objects somehow derive from relations, one still is committed to there being relations without relata in the fundamental physical domain in the first place.

As previously mentioned, the ontological primacy of relations has been spelled out in accordance with a weak notion of discernibility and a weak version of PII (Saunders, 2003), distinguishing objects in virtue of the symmetric and irreflexive relations holding between them. Still, this proposal turns out to be partially controversial in the structuralist literature; weak PII does not actually show how to derive objects from relations, since it can at best determine how many objects there are – ensuring an *epistemic* access to them – but not which one in which; a proper individuation of objects remains to be accomplished, distinguishing objects *qualitatively* and not just *numerically* (Ladyman and Bigaj, 2010). A similar conclusion is advanced by Dieks and Versteegh (2008, p. 926, quoted in Esfeld and Lam, 2011, p. 149): «because of the symmetry any property or relation that can be attributed to one object can equally be attributed to any other and we can therefore not single out any specific object». Even if we acknowledge that weak PII grounds the numerical diversity of objects, and nothing more, other worries arise: first, French (2010, p.105) notes that the resulting conception of objects is so *thin* that the distinction between *Priority-based OSR* and E*liminative OSR* appears significantly blurred. What distinguishes a thin notion of objects from no notion of objects at all, or what makes an object thin? According to French (pp. 105-106), *Priority-based OSR* understands thin objects as merely *conceptual* objects, with no clear physical correlates. Second, structures, in order to individuate the *relata*, seem to *presuppose* their numerical diversity, and then cannot account for it (MacBride, 2006).[24] Either way, we are back where we started, and the 'relation without *relata*' re-emerges at a different level: «without distinct individuals that are metaphysically prior to the relations, there is nothing to stand in the irreflexive relations that are supposed to confer individuality on the *relata*» (Ladyman, 2020, sec. 4.).

At the same time, assuming the numerical diversity of objects as primitive is troublesome for different reasons. This is a natural outcome of *Moderate OSR*: if objects and relations are on a par, then the numerical diversity of objects should be established in advance and independently of structures. However, the idea of a primitive numerical diversity easily collapses into those concepts of primitive thisness or haecceity the structuralist views want to contrast, for they suggest the priority of objects over relations. In this respect, Esfeld and Lam (2008, p. 33) insist that primitive numerical diversity is a quite different notion:

> A numerical distinction is not a primitive thisness, for it does not establish an identity in time—or any other sort of an identity—that is not empirically accessible. Accepting a numerical distinction as primitive is motivated by the physical cases—quantum entanglement, space-time points—in which there is a plurality of objects without these objects being distinguished from one another by any intrinsic properties or relations in which they stand and without primitive thisness being an open way out, since there are strong physical arguments against primitive thisness.

Despite this clarification, French (2010, p. 105) points out that *Moderate OSR* is still committed to a too thin notion of objects: claiming that objects are entirely defined by their relations amounts to saying that there are no objects at all – again, we have a version of the 'relation without *relata*' objection: relations require an independently grounded identity of objects in order to obtain.

Some of these criticisms are addressed by Esfeld and Lam (2011) who considerably revise their original moderate position (developed in Esfeld and Lam, 2008) outlining a conceptual – rather than an *ontological* – relationship between objects and structures. In particular, objects and re-

[24] Analogous considerations apply to the mathematical framework and meet with similar difficulties (see chapter 2, sec. 2.4.2.).

lations are one and the same thing and relations are the *modes* in which objects exist:[25] «In reality, there is only one type of entity, namely objects that exist in particular ways» (Esfeld and Lam, 2011, p. 151). In this framework, it is more plausible to say that objects and structures are given 'at once', for objects exist only as standing in the relations and structures are concrete exactly because they are ways in which physical objects exist. Of course, this conception involves a deep reconsideration of the nature of properties: the standard view of properties as universals instantiated by particular individuals is replaced by the idea of properties as *modes*/*tropes*, i.e. ways in which objects exist. Assuming that on this revised version of *Moderate OSR* tropes are relations, and not intrinsic properties, objects appear to be *bundles of relations*, thus introducing a further interpretation of objects as well.

In this context, the question of which version of OSR – if any – is correct is yet to be answered. Some reasons to favour non-eliminative interpretations of OSR will be provided in chapter 4, where I will show that the interpretation of the relationship between objects and structures in terms of ontological dependence and grounding naturally supports the introduction of objects in the structural ontology. In the same chapter, I will also propose a further interpretation of OSR and a related alternative strategy to avoid the 'relation without *relata*' objection. Interestingly, the debate on scientific OSR and its 'relation without *relata*' objection is intimately connected with the debate on mathematical structuralism, to which the next chapter is devoted.

[25] As specified by Esfeld and Lam (2011), this idea echoes Spinoza's *Ethics* (1677), in which the conception of properties as *modes* was originally proposed.

2. Individuating Objects in Mathematical Structuralism

In the present chapter, I will present mathematical structuralism and its main varieties, examining in particular how mathematical objects and their structural properties are individuated.
To begin with, following Hellman and Shapiro (2019), I will provide a brief historical background of the debate, focusing on some early structuralist ideas in Dedekind (1872; 1888). On this basis, I will provide a more precise characterization of the philosophical core of mathematical structuralism by referring to Benaceraff (1965), which opened the way to a more recent and explicit development of structuralist claims in the philosophy of mathematics.

Along the lines of Reck and Price (2000), Parsons (1990) and Hellman and Shapiro (2019), some important taxonomic distinctions will be drawn, i.e. those between methodological/philosophical positions and eliminative/non-eliminative forms of structuralism; within non-eliminative structuralism, *in re* and *ante rem*/*sui generis* structuralism will be also distinguished. With regard to the first distinction, I will focus on the philosophical views and illustrate the relative, universalist and pattern forms of structuralism as a general framework for mathematical structuralism. Concerning the second distinction, I will refer to Hellman's (1989) eliminative modal account and Shapiro's (1997) non-eliminative *ante rem* approach (as opposed to set-theoretic *in re* structuralism) respectively.

I will then assume Shapiro's (1997) view as a more promising version of structuralism, whose conception of mathematical objects can be compared with the interpretation of physical objects in Ontic Structural Realism (OSR) illustrated in chapter 1. In particular, I will draw a comparison between *ante rem* structuralism and scientific *Priority-based OSR,* which admit objects in the ontology and yet define them in purely structural terms.

The second part of the chapter highlights one of the most critical aspect of *ante rem* structuralism, i.e. the *identity problem* concerning objects in structures with non-trivial automorphisms, as appealed to in Burgess (1999) and Keränen (2001). As before, this criticism recalls the 'relation without *relata'* objection concerning scientific OSR. Lastly, different responses to the identity problem (Ladyman 2005; MacBride 2006; Shapiro 2006a; 2006b; 2008; Ladyman & Leitgeb, 2008) will be discussed, evaluating their pros and cons.

2.1. Mathematical structuralism: the state of the art

The theoretical core of mathematical structuralism can be adequately presented by considering the following quotation from Hellman and Shapiro (2019, p.1):

> The theme of structuralism is that what matters to a mathematical theory is not the internal nature of its objects – numbers, functions, functionals, points, regions, sets, etc. – but how these objects relate to each other.

A similar way of expressing the same idea is the following:

> Mathematics is concerned principally with the investigation of structures of various types in the complete abstraction from the nature of individual objects making up the structures (Hellman, 1989, vii).

According to Reck and Price (2000, pp. 341-342), these intuitive assumptions correspond to three structuralist theses:

> (1) that mathematics is primarily concerned with "the investigation of structures"; (2) that this involves an "abstraction from the nature of individual objects"; or even, (3) that mathematical objects "have no more to them than can be expressed in terms of the basic relations of the structure".

For short, let us re-label theses (1)-(3) as follows:

(1) Structures
(2) Abstraction
(3) Relations

Theses (1)-(3) offer some promising guidelines to briefly reconstruct the historical background of mathematical structuralism – specifically focusing on the role of Dedekind (1872; 1888) as a forefather of a structuralist perspective on arithmetic.

2.1.1. Historical Background

As pointed out by Hellman and Shapiro (2019, chapter 2) the early origins of mathematical structuralism trace back to some significant transformations in mathematics in the 19th century, with the progressive abandonment of the role of (Kantian) *intuition* and the need for understanding the *a priori* and the necessity of mathematics in purely *formal* terms.

Such change was largely motivated by the emergence of the axiomatic method and model-theory and by the introduction of new *ideal elements* in both geometry and arithmetic (i.e. imaginary points and complex numbers respectively).[26] This resulted in a significant 'move away' from intuition and in a stronger focus on formal and abstract features of mathematical theories, opening the way to a structuralist understanding of them.

In geometry, a new interpretation of the discipline came up: in contraposition with the standard idea of a *real* and *applied* science, investigating matter and extension, geometry began to be considered a *formal* and *rigorous* science, independent of the intuitive content of the theory; Grassman (1844), Hilbert (1899) and Poincarè (1908) marked the structuralist turn of geometry and led to its interpretation as the *science of pure forms* (Grassman, 1844). On this view, geometry is concerned with *formal structures* and *abstracts away* from the internal nature of geometrical terms, defined solely with respect to the *relations* between them – thus showing a first commitment to the structuralist theses (1)-(3).[27]
Hilbert's (1899) *Grundlagen Der Geometrie* represented the culmination of this structuralist trend, in which the logical relations between ideas completely replaced spatial intuitions:[28]

> We think of . . . points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated," "between," "parallel," "congruent," "continuous," etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry*.

---

[26] Hellman and Shapiro (2019, pp. 9-10) mention three main ways of introducing these new entities in mathematics: postulation, implicit definition and construction.

[27] See Grassman (1972, p. 47, quoted in Hellman and Shapiro, 2019, p.11): «no meaning is assigned to an element other than that. It is completely irrelevant what sort of specialization an element really is…it is also irrelevant in what respect one element differs from another, for it is specified simply as being different, without assigning a real content to the difference».

[28] Intuitions preserved a role in the source of geometrical axioms but, as noticed by Hellman and Shapiro (2019, p. 18), «Once the axioms have been formulated, intuition is banished».

As a natural outcome of the 'banishment' of intuitions, any interpretation of a theory was fixed up to *implicit definitions* and *isomorphisms*. First, implicit definitions are set of sentences that specify the meaning of a newly introduced term *contextually*, i.e. as a function of the meaning of the larger expressions in which they occur.[29] Second, an isomorphism is a one-to-one function from the domain of one model to the domain of another model, in which all the relevant relations are preserved. Shapiro (1997, p. 91) presents isomorphisms as follows:[30]

> Suppose, for example, that the first system has a binary relation *R*. If *f* is the correspondence, then *f*(*R*) is a binary relation of the second system and, for any objects *m*, *n*, of the first system, *R* holds between m and n in the first system if and only if *f*(*R*) holds between *f*(*m*) and *f*(*n*) in the second system. Informally, it is sometimes said that isomorphism "preserves structure".

Consider a more formal definition of isomorphic systems:

> Two systems $S = (R_1..., R_n)$ and $S'= (R'_1 ..., R'_n)$ are isomorphic ($S \cong S'$) iff there is bijective function $f : D \rightarrow D'$ such that if R_i is a k-ary relation in S, then $(\forall x_1 ..., x_k \in D)$ $[R_i (x_1 ..., x_k)$ $R'_i (f(x_1)..., f(x_k))]$. (Schiemer and Wigglesworth, 2019, p. 1207).

The isomorphism between systems is an equivalence relation that is standardly used to define criteria of identity for abstract structures: taking [S] to be the abstract structure of the system S, the following Fregean abstraction principle obtains:

---

[29] See Antonelli (1998, p. 151). Implicit definitions play a prominent role in modern mathematics and especially in formal axiomatics. They are also crucial in several philosophical debates, i.e. in neologicism (Hale and Wright, 2001) and in mathematical structuralism (Shapiro, 1997).

[30] The notion of isomorphism should be distinguished from that of automorphism, which is significant for the discussion on the identity problem affecting *ante rem* structuralism (cf. section 2.3). An automorphism is an *inner* isomorphism, i.e. an isomorphism from the structure to itself preserving the internal relations. An automorphism is trivial when it is just given by the relation of identity defined on every structure; structures with trivial automorphism (such as the natural numbers structure) are called *rigid structures*. By contrast, an automorphism is non-trivial when it is not an identity mapping: consider for example the change of sign in the complex numbers structure, mapping each number with its conjugate. Structures with non-trivial automorphisms (i.e. the relative numbers structure, the complex numbers structure, the Euclidean plane, etc.) are defined *non-rigid* structures.

(1) $[S] = [S'] \longleftrightarrow S \cong S'$

Defining theories by means of implicit definitions and isomorphism, the derivation of theorems was no longer dependent upon any specific interpretation of the geometrical elements, but just upon their stipulated relations.

Let us now investigate the historical background of a structuralist approach in arithmetic, that is more relevant for the present discussion. Dedekind's (1888) *Was sind und was sollen die Zahlen?* plays a paradigmatic role in the foundations of mathematical structuralism. Specifically, the following definition introducing natural numbers constitutes a manifesto of structuralism:

> 73. Definition. If in the consideration of a simply infinite system N set in order by a function ϕ we entirely neglect the special character of the elements, merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. […] With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions […] are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements. (Dedekind, 1888, p. 68, original emphasis).

In order to cash out the structuralist import of this passage, some notions stand in need of clarification; first, Dedekind defines 'infinite system N' as follows:

> 71. Definition. A system N is said to be *simply infinite* when there exists a similar function ϕ of N in itself such that N appears as chain (44) of an element not contained in ϕ (N). We call this element, which we shall denote in what follows by the symbol 1, the *base-element* of N and say that the simply infinite system N is *set in order* by this function ϕ.

In other words, a set S and a function ϕ are a 'simply infinite system' if ϕ is a one-to-one function, there is an element *e* that is not in the range of S and the only subset of S that both includes *e* and is closed under ϕ is S itself. The symbols 'N' and '1' suggest that Dedekind understands the simple infinite system N as a model of the natural numbers, closed under the successor function.

Dedekind implicitly takes simply infinite systems to be *isomorphic*: «all simply infinite systems are similar (i.e. isomorphic) to the number series N and consequently [...] also to one another» (1888, sec. 132). This provides a structuralist explanation for the last sentence of definition (76), claiming that «the relations or laws which are derived entirely from the conditions [...] are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements».

The phrases «neglecting the special character of the elements», «freeing the elements from every other content» – defining Dedekind's *abstraction* – and «free creation of the human mind» – introducing Dedekind's notion of *creation* – deserve a special attention. Reck (2003) observes that several conceptions of abstraction and creation can be put forward, depending on which interpretation of Dedekind's structuralism is endorsed.[31] In principle, Dedekind's structuralism is consistent with different forms of structuralism, i.e. specifically, set-theoretic structuralism and *ante rem/sui generis* structuralism, which will be illustrated in the following sections. Despite some interesting connections with these views, Reck (2003, sec. 11) attributes to Dedekind a form of *logical structuralism,* which is strictly connected with the mathematical methodology and still neutral with respect to more specific philosophical theses.
This perspective introduces a precise understanding of Dedekind's abstraction and creation. With respect to abstraction, the following considerations are put forward:

> After having constructed a simple infinity, Dedekind tells us to "abstract away" from all the non-arithmetic properties of the objects in it. All that matters, instead, are the arithmetic truths that hold in it, i.e., those truths expressible in the particular language specified by him [...](Reck pp. 399–400).

In this process, arithmetical truths are those that *logically* follow from the basic principles. Consequently, a logical notion of abstraction emerges, in virtue of which the properties that are abstracted away are the non-arithmetical properties of objects which cannot be expressed in the language of the simply infinite system, i.e. the properties that cannot be defined in terms of the fundamental notions. This strategy

---

[31] For example, on a set-theoretic reading of Dedekind's structuralism, «neglecting the special character of the elements» and «freeing the elements from every other content» seem to refer to a sense of abstraction in terms of 'putting aside', 'ignoring' the non-structural properties of objects insofar they are considered as 'objects of *our* investigation'. However, the same interpretation appears quite weak if applied to the notion of creation: «what gets created are merely certain new objects of study, i.e. objects as seen or investigated by us, not objects in themselves?» (Reck, 2003, p. 382). Other problems arise for the psychologist understandings of Dedekind's claims, according to which both abstraction and creation would denote psychological processes whose result are merely *mental* mathematical entities. In fact, the idea of mental entities requires an owner to be specified, assuming that different people's models of numbers are somehow isomorphic to each other - something that is hard to establish.

is motivated by the alleged isomorphism between different simply infinite systems, in virtue of which all of them satisfy the same arithmetical truths. As claimed by Hellman and Shapiro (2019, p. 30) «if two systems are isomorphic, then the structures obtained from them by Dedekind abstraction are isomorphic».

Such interpretation of abstraction sheds light on the the concept of creation as well: to say that «numbers are a free creation of a human mind» means that a new system of objects is introduced by abstraction and that this system does not coincide with any of the simply infinite systems previously created. To put this in Cantor's (1883) terminology, to create a simply infinite system amounts to uniquely define a certain *conceptual possibility,* i.e. a particular simply infinite system. At this stage, the question concerning *which system is the natural numbers system* arises: «it is that simple infinity whose objects only have arithmetic properties, not any of the additional, "foreign" properties objects in other simple infinities have» (Reck, 2003 p. 400). On these grounds, the created natural numbers correspond to a single collection of objects, i.e. the abstract type of a simple infinite system. All this considered, Dedekind (1888) appears to treat natural numbers as *sui generis* objects – which are different from both ordinary objects and from objects in other simply infinite systems – in close analogy with *ante rem/sui generis* structuralism, which will be explored further on.

Some remarks from Dedekind's (1872) *Stetigkeit und Irrationale Zahlen,* including the well-known account of continuity and real numbers, make the interpretation of creation more precise, clarifying that what is actually created is a *new* set of objects.

Dedekind (1872) observes that rational numbers can be mapped one-to-one in any straight line, assuming a point on the line as the origin and an interval as a unit. Rational numbers are not *continuous*: there are points on the line which do not correspond to any rational number. However, we lack an *intuitive* understanding of what continuity is; we can just *attribute* continuity to a line. Dedekind (1872) aims at filling these gaps by introducing the notion of *cuts*; Dedekind defines a *cut* to be a division of the rational numbers into two non-empty sets (A_1, A_2), such that every member of A_1 is less than any member of A_2. Because of the discontinuity of the rational numbers, there are some cuts which are not produced by any rational number:

> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new *irrational* number a, which we regard as a completely defined by this cut (A_1, A_2); we shall say that the number a corresponds to this cut, or that it produces this cut. (Dedekind, 1872, § 4).

Note that Dedekind emphasizes that the created real numbers are not *identified* with the cuts, but just *correspond* to them: this shows that Dedekind introduces *new* numbers, different from the cuts themselves. The identification between the reals and the cuts is rejected because the cuts display some properties which would appear odd if applied to the corresponding real numbers (cf. Hellman and Shapiro, 2019, p. 29). So, once again, *creation* appears as a result of an abstraction process: «the whole process is seen as involving a process of *abstraction* in the sense of ignoring all the additional properties that cuts have when using them as real numbers» (Reck, 2003, p. 384).

Once abstraction and creation in Dedekind (1872; 1888) have been clarified, the structuralist insight of Dedekind's (1888) definition (76) becomes more evident. First, what Dedekind defines is the *structure* of the natural numbers as *a whole,* since the language and the arithmetical truths for natural numbers are determined *together*. Second, the process of *abstraction* ensures that the resulting natural numbers system is considered independently of ('abstracting from') their 'foreign', non-arithmetical properties such as, for instance, for the number 9, being the number of planets in our Solar System or, for the number 2, being an element of the set {1, 2, 3}. This is because such properties appear to be not *constitutive*, i.e. not tied to the identity of an object. Arguably, non-constitutive, non-arithmetical properties are those properties that an object displays in *isolation*: «Yes, the number 9 has the property of being the number of planets in the Solar System. It is just not a constitutive property; the number 9 would still be the particular number it is even if there was an additional planet beyond Pluto» (Reck, 2003, p. 408).

Lastly, the following sentence from definition (76) «[…]merely retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function φ […]» suggests that the nature of natural numbers is fixed by the *relations* between them, which determine the arithmetical, constitutive properties of natural numbers, i.e. for the number 3, being the successor of the number 2.

On this basis, the theses (1)-(3) outlined by Reck and Price (2001) – (1) Structures, (2) Abstraction and (3) Relations above – are satisfied by Dedekind (1888) and spell out the precise sense in which Dedekind's position amounts to a form of structuralism – which, though, still waits for a deeper philosophical interpretation.

2.1.2. Benacerraf's challenge

A more explicit *philosophical* commitment to theses (1)-(3) firstly appeared in the 1960s', when Benacerraf (1965) and Putnam (1967) introduced structuralist claims in the philosophy of mathematics.

In this context, I will specifically deal with Benacerraf's "What Numbers Could not Be" (1965), since it plays a more significant role in the mathematical structuralism debate.

This paper presupposes the interpretation of mathematics as founded by axiomatic set-theory, in which numbers are identified with sets. However, Benacerraf formulates a serious challenge to set-theoretic platonism, which is based on the following argument: let us imagine two children, Ernie and Johnny, who have learned logic – set-theory in particular – instead of arithmetic. Significantly, for Ernie and Johnny, the Peano-Dedekind axioms (PA_2) correctly refer to sets and sets are by themselves sufficient to (explicitly) define and use the arithmetical operations, including counting mathematical objects: any set has *k* members if it can be put in a one-to-one correspondence with the set of numbers less than or equal to *k*. On this basis, learning numbers would just require associating new names to familiar sets; in other words, «old (set-theoretic) truths took on new (number-theoretic) clothing» (Benacerraf, 1965, p. 48). An important feature of sets is that of belonging to other sets; thus, if numbers are just sets, it is legitimate to raise the (extra-arithmetical) question concerning whether a number belongs to another number – say, whether 3 belongs to 17. The main problem lies in the fact that Ernie and Johnny understand numbers in terms of different interpretations of sets: the Von Neumann's and the Zermelo's ordinals, which are characterized as follows.

Von Neumann:

0 = Ø

1 = {Ø}

2 = {Ø, {Ø}}

3 = {Ø, {Ø}, { Ø { Ø}}

Zermelo:

0 = Ø

1 = {Ø}

2 = {{Ø}}

3 = {{{Ø}}}

The two accounts refer to different cardinality relations: though they agree that a set has *n* members if and only if it could be put into one-to-one correspondence with the set of numbers less than or equal to *n*, the Von Neumann's theory also assumes that a set has *n* members if and only if it can be put into one-to-one correspondence *with the number n itself*. On that view, the number 17 has 17 members. Nevertheless, this thesis is rejected on the Zermelo's conception, where 17 is deemed single-membered. This means that in Von Neumann's theoretical framework it is plausible to say that 3 belongs to 17, but this is clearly not the case in Zermelo's. Therefore, the two accounts often provide diverging responses to the relevant question, i.e. does a given number belong to another given number? However, we have no compelling reasons to privilege one over the other: both Zermelo's and Von Neumann's characterizations of sets satisfy the conditions for natural numbers. Still, the two accounts cannot be both right, for «if the numbers constitute one particular set of sets, and not another, then there must be arguments to indicate which.» (Benacerraf, 1965, p. 58). Since arithmetic cannot supply those arguments, neither Zermelo's, nor Von Neumann's account of sets can be actually accepted as the correct one – «exactly one is correct or none is» (p. 57). Consequently, natural numbers are not *sets*, after all.

Benacerraf starts from this basic idea to draw a more radical conclusion, according to which not only natural numbers are not *set-theoretic objects*, but they are no *objects* at all. In fact, as we have no reasons to state which sets numbers correspond to, we have no more reasons to identify a number with a particular object or another. In both cases, what really matters is not the intrinsic nature of the sets or objects involved, but that they are arranged in an *appropriate progression*. Potentially, any sets or objects which constitute an appropriate progression adequately display the relevant properties of the natural numbers because «what is important is not the individuality of each element but the structure which they jointly exhibit» (p. 69). The abstract structure is exactly what Zermelo's and Von Neumann's set-theoretic systems *share* merely in virtue of being progressions, despite their differences concerning which particular sets or objects instantiate this structure.

This explicitly introduces a structuralist approach, in which numbers turn out to be *positions* – not the specific objects occupying them. Positions are completely reduced to their *structural properties*, that Benacerraf defines as the properties deriving from the relations they bear to one another in virtue of being arranged in these progressions: «mathematical objects have no properties other than those relating them to other "elements" of the same structure» (p. 70).

Nonetheless, the abstract structures at play, the positions in these structures and their structural properties are still somehow vague and in need of further clarification.
To make these notions more precise, it is useful to appeal to Reck & Price's (2000) distinction between a methodological structuralist stance and its philosophical developments.

2.1.3. Methodological *vs* philosophical structuralism

Reck and Price's (2000) purpose is to explore different varieties of structuralism in the philosophy of mathematics and the ways in which they are related to each other. In particular, they distinguish a *methodological* structuralism[32] – strictly connected with the mathematical practice and philosophically neutral – from those proposals that investigate the semantical as well as the metaphysical issues of structuralism more deeply. In this context, I will specifically take into account the latter, which properly introduce the varieties of structuralism that will be crucial in the present discussion. In a philosophical perspective, the *relativist*, *universalist* and *pattern* forms of structuralism can be distinguished. As I am going to explain, these positions entail different interpretations of the theses (1)-(3) illustrated above and of the notion of *structuralist abstraction*.

Before moving to the three structuralist approaches and their main distinctions, some preliminary remarks are needed. First, we should define the notions of *system* and *structure*: to be a *system*, is to be «a collection of objects together with certain relations on these objects» (Hellman and Shapiro, 2019, p. 1). Consider as a standard example the natural numbers system, corresponding to an infinite collection of objects with an initial object and a one-to-one successor function satisfying the axioms of second-order arithmetic. Once systems are defined, a *structure* is understood as the «abstract form of a system, which ignores or abstracts away from any features of the objects that do not bear on the relations» (*ibid*., p. 2). Therefore, the structure of the natural numbers is a sort of universal, a *one over many* shared by all the natural numbers systems. Second, we should shed light on the notion of *model*, which plays a crucial role in relativist structuralism. A model of a given theory is composed by a set or class (the domain of the model) and the relations, functions or operations ranging on that domain, which provide an interpretation of the primitive terms of the theory. The related notion of *satisfaction* of formulae accounts for an assignment of relations and

[32] Corresponding to early structuralist ideas presented by Hilbert (1899) and Dedekind (1888).

operations to the primitive symbols of the given theory, such that a unique truth-value is determined for each sentence of the language.

With these premises, let us begin with relativist structuralism and its main features. Relativist structuralism starts from the standard set-theoretic conception of arithmetic (for this reason, it is also defined *set-theoretic structuralism*) where the Peano-Dedekind axioms (PA_2) are formulated in second-order logic and the two non-logical symbols 1 (individual constant) and *s* (successor function) define all the other mathematical symbols. Nevertheless, this position denies that the reduction of numbers to sets can be understood in any absolute sense. Indeed, one can arbitrarily pick any model *M* of PA_2, since what we identify as the natural numbers entirely depends on this initial and provisional choice, whose reference is fixed until another model is selected. The notion of truth is expressed in a pragmatic and relativistic way as well, since all that matters is *consistency* with the relevant framework. However, the definition of both reference and truth appears to be unproblematic in relativist structuralism, since all models of PA_2 are *isomorphic*; in this respect, Reck and Price (2000, p. 350) claim that «[…] while truth has been defined in a relative way, a non-relative notion of "truth in arithmetic" is actually implied: truth in all models of PA_2 ». This idea also clarifies a first interpretation of *structural abstraction*: when we initially pick a model, we completely abstract from the peculiar properties of the objects within that model, in the sense of *ignoring* them, i.e. non considering them in *our* investigation.

On the one hand, relativist structuralism represents a form of *eliminative structuralism*; Reck and Price (pp. 353-354) mention two main reasons for holding this interpretation. First, there is no appeal to any *unique* or *special* system of the natural numbers – any model of PA_2 can play the role of the natural numbers system, the real numbers system, etc., and none of them is privileged. In this respect, we should not consider any of these models as the natural number system; there are no natural numbers whatsoever, in accordance with Benacerraf (1965). Secondly, mathematical structures themselves are conceived in purely set-theoretic terms, i.e. as isomorphic types within the set-theoretic hierarchy.

On the other hand, this is not to deny that abstract structures exist *at all*; relativist structuralism is in principle consistent with a form of *in re* Aristotelian realism, in which structures exist but are ontologically posterior to the systems instantiating them (cf. Hellman and Shapiro, 2019, p. 2). I will go back to this issue in the next section, when comparing relativist/set-theoretic structuralism to *ante rem* structuralism.

From the point of view of the mathematical practice, relativist structuralism turns out to be quite a natural approach, as it allows mathematicians to conceptualize different branches of mathematics (arithmetic, analysis, group theory, etc.) in a unified, set-theoretic perspective. Still, sets are understood as entities of a special kind, and this seems to suggest a realist or a platonist approach, conflicting with the overall structuralist framework:

> Also, what is so special about sets that they deserve to be treated differently, i.e., granted some special, non-structuralist kind of reality? Put the other way around, if we can treat sets that way, why not the natural numbers, the real numbers, and other mathematical objects as well? (Reck and Price, 2001, p. 352).

Let us now turn to the *universalist* variety of structuralism. Universalism interprets mathematical statements as referring not to *any* model of PA_2, but to *all* these models *taken at the same time*. Mathematical statements are treated as universal if-then statements, that universally quantify over the relevant systems by referring to *all* models*, all* one-place predicates and *all* one-place functions altogether. This introduces a different sense of structural abstraction: the specific features of the single models are abstracted away through a *generalization* process.

Universalist structuralism can be seen more clearly as a form of eliminative structuralism, because it avoids a commitment to unique and special models (whose specific features are not here *ignored* as objects of our investigation, but *generalized* in the universal if-then statements) and to structures as objects in general (no extra mathematical structure is added over and above the relational systems, even as ontologically posterior to them). This position is affected by the so-called *non-vacuity problem*, according to which the universal if-then statements are vacuously true (including, for example, if PA_2, then 1+ 1= 0) since there is nothing that satisfies the antecedent, i.e. if there are no models of PA_2. This concern has led to different proposals: some refer to set theory as providing the needed models (along the lines of relativist structuralism) whereas others – such as the modal and nominalist varieties of universalism – invoke a modal turn.

Finally, *pattern structuralism* argues that the PA_2 models – being isomorphic – exemplify the same abstract structure or *pattern*. Significantly, the notion of pattern constitutes the notion of *a new kind of abstract entity*, which is fundamentally different from the relational systems corresponding to it. In this respect, Reck and Price (2000, p. 363) claim that «what we really study in arithmetic, in the end, are not the various particular models of PA_2, but something in addition to them: a corresponding *pattern*». For this reason, pattern structuralism consists in *a non-eliminative* form of structuralism: firstly, it clearly does not do without abstract objects; new abstract entities, the patterns, are posited over and above the set-theoretic relational systems. Secondly, it is committed to a special and unique structure of natural numbers – precisely, the natural numbers pattern. On the one hand, a pattern represents a universal, since each pattern has different exemplifications; for example, the natural number pattern is instantiated by both the Zermelo and the Von Neumann's ordinals. On the other hand, as pointed out by Hellman and Shapiro (2019, p. 2) they are distinguished from traditional universal, such as properties, because «[...] a given property applies to, or holds of, individual objects, while a given structure applies to, or holds of, entire *systems».*[33]

2.1.4. Eliminative *vs* non-eliminative structuralism

Reck and Price's (2000) taxonomy of relativist, universalist and pattern structuralism allows us to more precisely draw the distinction between *eliminative* and *non-eliminative* structuralism. This dichotomy, explicitly introduced by Parsons (1990), distinguishes a structuralism *without structures* from a structuralism *with structures.* [34] On the one hand, the eliminative side, relying on universalist structuralism, rejects the existence of structures *at all*: mathematical statements do not actually single out any specific system of natural numbers. By contrast, they are generalizations over *all* systems of objects instantiating a structure and they just require that the relations and functions of a given domain satisfy the relevant conditions. Consequently, the reference to both particular objects and the abstract structure is simply avoided or eliminated – any reference to the structure is just a convenient tool to denote all the relevant isomorphic systems. In the structuralist literature, eliminative structuralism has largely inspired (but is not reduced to) Hellman's (1989) modal structuralism.[35] Hellman's basic idea is that mathematical statements universally quantifying over *systems* can be replaced by statements universally quantifying over *possible* structures. In this way, mathematical statements can be non-vacuously true even if there are no systems of objects

[33] However, this last point is contentious: after all, systems work as individuals when patterns apply to them, and universals can be in principle applied to objects which are internally structured.

[34] This terminology has been later introduced by Hellman (1996). Note also that the distinction between eliminative and non-eliminative views is differently understood in the debate on scientific OSR, where it distinguishes a structuralism *without objects* from a structuralism *with objects.*

[35] For example, Chihara (2004) has articulated a different form of eliminative structuralism.

instantiating the structure – even if no *actual structures* exist: all that is required is the merely possible existence of structures. Such proposal meets with serious difficulties, first of all the ambiguity concerning the notion of modality adopted. Parsons discusses (2008, sec. 3.15) the logical, physical, metaphysical and mathematical interpretations of modality. Since all these interpretations raise problems for modal structuralism, as Hellman (1996) himself acknowledges, Hellman proposes to endorse a primitive conception of modality, that is even more controversial – especially as far as the task of clarifying an appropriate epistemology for modal structuralism is concerned (see Hale, 1996).

On the other hand, non-eliminative structuralism leaves room for a further distinction between *in re* structuralism and *ante rem* structuralism, based on relativist and pattern structuralism respectively. *In re* and *ante rem* varieties of non-eliminative structuralism do no longer engage in eliminating mathematical entities and share a general commitment to the existence of structures. As pointed out by Wigglesworth (2018, p. 224) what distinguishes the two positions is the *dependence relation* among structures and the systems instantiating them:

> […] the core thesis of *in re* structuralism is a dependence claim: for all abstract structures S, the existence of S depends on the existence of some system exemplifying the structure S. This dependence thesis differentiates *in re* structuralism from *ante rem* structuralism, according to which no such dependence between structures and systems holds.

In re structuralism corresponds to set-theoretic structuralism – or, in Reck and Price's (2000) terminology, to relativist structuralism – which traces back to the early origins of mathematical structuralism in the 1960s. As mentioned before, the core of this structuralist approach consists in an interpretation of structures in purely set-theoretical terms; on this view, structures are not eliminated *tout court* as *objects* – as it happens in the eliminative versions of structuralism. Still, the existence of structures as *sui generis* entities is denied, and that motivates the label *in re* denoting this form of structuralism. Structures are nothing over and above the sets instantiating them, so that there is not a *sui generis* (e.g. a unique) structure of the natural numbers, of the reals numbers, etc.: mathematical structures cannot exist without systems existing as well. In other words, a (reductive) dependence claim between abstract structures and set-theoretic systems holds.

From the semantical point of view, this means that mathematical terms can be treated at face-value provided that the adopted mathematical theory is set-theory itself; if other mathematical theories are at place, then the relevant terms are to be translated in set-theoretical terms – a revisionist semantical approach is introduced. Moreover, multiple representations of sets are available (i.e. Von Neumann and Zermelo's interpretations), thus leading to the Benaceraff's (1965) problem. On the contrary, *ante rem* structuralism assumes *background structures* existing *independently of* and *prior to* the systems instantiating them: structures are *sui generis* entities and,[36] as such, no dependence relation between them and the corresponding systems obtains. This allows mathematical statements to be interpreted at *face value*, since they do not generalize over all systems of objects but exactly refer to the positions within the natural numbers, the real numbers structure, etc. In this picture, mathematical objects are entirely defined by their structural properties and their relations with the other objects in the same structure.

Shapiro (1997) and Resnik (1997) have developed very similar accounts of *ante rem* structuralism. In particular, they both assume that mathematical structures are patterns existing independently of their concrete instantiations. Moreover, they both interpret mathematical statements as mind-independently true or false and assign a crucial role to the notion of isomorphism, in virtue of which any structure can be instantiated by a class of isomorphic systems.[37]

Still, the two positions differ for a number of reasons; first, Shapiro is more explicitly committed to semantical as well as metaphysical issues and elaborates a genuine *theory of structures*, which is more full-fledged and more realist in character than Resnik's account – in which structuralism is understood as quite independent of the realist/anti-realist debate. Second, while Resnik endorses an epistemology that relies on some Quinean intuitions,[38] Shapiro outlines an original epistemological system which introduces concepts like those of *abstraction* and *implicit definition*. Third, in Shapiro's position, a more detailed discussion of both abstract structures and the positions within them is provided.

It is worth to observe that other forms of mathematical structuralism, going beyond the present taxonomy, have been recently developed and deserve some serious attention; among them,

---

[36] For this reason, Hellman and Shapiro (2018) label *ante rem* structuralism *sui generis* structuralism.

[37] Actually, these intuitions precisely concern non-algebraic structures, whose relational systems are *categorical*, i.e. isomorphic to each other. However, not all mathematical theories display this feature (for example, group theory inclu-des non-isomorphic relational systems) and Shapiro and Resnik suggest dealing with them in a more derivative way.

[38] In particular, Resnik (1997) refers to indispensability considerations, and to the familiar Quinean doctrines of holism and naturalism, though providing significant refinements and clarifications of them in the framework of mathematical structuralism.

category structuralism (Awodey, 1996), modal set-theoretic structuralism (Linnebo, 2013) and abstractionist forms of structuralism (Linnebo and Pettigrew, 2014) play a prominent role in the mathematical structuralism debate. Still, they are not strictly related to the present discussion and their investigation is a topic for another work (see Reck and Schiemer, 2020, sec. 3.1., 4., and Hellman and Shapiro, 2019, for a more comprehensive analysis).

In the remainder of this chapter, I will specifically explore Shapiro's (1997) non-eliminative *ante rem* structuralism, which offers a deep – and intuitively compelling – insight into mathematical structuralism. However, even this position is not exempt from serious challenges (i.e. the identity problem) which will be discussed in detail in section 2.3.

2.2. Shapiro's non-eliminative *ante rem* structuralism

Shapiro aims at introducing a structuralist position which combines realism in ontology (mathematical entities exist) and realism in semantics (mathematical statements have not-vacuous truth values) with an acceptable epistemology, thus responding to the so-called Benacerraf's dilemma (1973). According to Benacerraf, the semantical and the epistemological *desiderata* are inconsistent. On the one hand, ontological realism embraces a convincing semantics, whereby mathematical statements are interpreted at face-value. Still, the abstract nature of these objects – which makes them not located in space-time and not causally effective – introduces serious epistemological problems.[39] On the other hand, ontological anti-realism about objects ensures a more straightforward epistemology but cannot account for a corresponding semantics, which would require objects to refer to.

The task of reconciling the epistemological and the semantical requirements is performed by providing a more precise definition of mathematical structures and the positions within them.
I will firstly present Shapiro's conception of mathematical structures, which are posited in such a way that is supposed to avoid the epistemic access problem.

[39] Especially because in this context Benacerraf (1973, pp. 671-673) presupposes a causal theory of knowledge (cf. Goldman, 1967): «I favor a causal account of knowledge on which for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S. [....] [But] [...] combining this view of knowledge with the "standard" view of mathematical truth makes it difficult to see how mathematical knowledge is possible. [...]»

2.2.1. Shapiro's theory of structures

The core of Shapiro's view consists of the label *ante rem* that characterizes this form of structuralism: structures exist independently of and prior to the concrete systems instantiating them. In this perspective, abstract structures behave as *universals*, which will not cease to exist if the corresponding concrete systems are removed. At the same time, structures may be treated as *particular objec*ts in themselves, which display existential and epistemological conditions. Such conditions are laid down according to a novel axiomatic theory of structures (Shapiro, 1997, pp. 95-5). In particular, this theory specifies eight axioms for the existence of structures. The axiom of infinity (1) posits the existence of structures with infinite places; the axioms of abstraction (2), subclass (3) and reduction (4) state that given an initial structure, infinite others may be obtained by adding or reducing places; according to the axiom of power-structure (5), if there is a structure, then another structure exists with as many places as there are subsets of the former one; the axiom of replacement (6) establishes that for any function that maps each place in a structure $\sum$ with each place in another structure $\sum_x$, there is a structure that is as large as the set-theoretic union of the places of both structures. The last two axioms illustrate existential conditions for structures in more general terms. The axiom of coherence (7) claims that if *k* is a coherent set of formulae expressed in a second-order language, then a theory satisfies *k* (i.e. it describes a single structure or different structures which are isomorphic). In other words, «any class of second order sentences that is coherent characterizes at least one *ante rem* structure» (Hellman and Shapiro, 2019, p. 53). The axiom of reflexivity (8), finally, states that the first seven axioms hold provided that *k* is a coherent body of formulae in the theory of structures. It is important to notice that the notion of *coherence*, which plays a crucial role in Shapiro's analysis, should be distinguished from the notion of *consistency* (no contradiction is derivable from *k*) and rather defined in an informal and primitive way.[40] However, the reference to coherence as a criterion for the existence of structures has been considered highly questionable, and in principle no clearer than Hellman's (1989) modal notion of mere possibility.

Shapiro's theory of structures is apparently modelled on set-theory, but justified on different grounds, as the set-theoretic terminology can be replaced by a not set-theoretic second order

[40] This is explained by two main reasons: firstly, a consistent theory does not necessarily entail *completeness* at the second order, and then it is possible to have consistent but not satisfiable theories. Secondly, consistence, in order to deny that certain derivations are possible, is generally based on modal notions, and this would require resources which are external to theory of structures defined by Shapiro (1997).

language. In fact, Shapiro suggests that structures are endowed with an intrinsic explanatory power that would enable them to justify and explain themselves, with no need of extra-structural resources:

> In short, on any structuralist program, *some* background theory is needed. The present options are set theory, modal model theory, and *ante rem* structure theory. The fact that any of a number of background theories will do is a reason to adopt the program of *ante rem* structuralism. *Ante rem* structuralism is more perspicuous in that the background is, in a sense, minimal. On this option, we need not assume any more about the background ontology of mathematics than is required by structuralism itself. (Shapiro, 1997, p. 96).

This conclusion – which is related to the more radical claim that mathematics accounts for itself – is largely controversial and does not adequately tackle Benacerraf's (1973) dilemma.

Still, Shapiro more specifically deals with the 'problem of access' by individuating three main ways in which mathematical structures can be known, depending on the nature of the structures at play. Firstly, small, finite structures can be abstracted through a process of *pattern recognition* from the concrete systems instantiating them.[41] Secondly, larger, finite structures can be known by both (indefinitely) extending the first method and defining equivalence classes among already given objects, so as to obtain a new structure. Lastly, infinite complex structures – to which any extension of pattern recognition cannot be applied – are solely accessible *via* implicit definitions, introduced by the seventh axiom illustrated so far and mainly focused on the notion of coherence. These different strategies help defining a stratified epistemological system which overall seems to better respond to Benacerraf's epistemic challenge.

Besides providing existential and epistemological conditions of structures, an *ante rem* theory of structures requires that *identity conditions* for structures are specified as well: «if we are to have a theory of structures, we need an identity relation on them» (Shapiro, 1997, p. 93).[42] Shapiro takes the identity among structures to be primitive and to be largely determined by the

[41] For a slightly different understanding of patter recognition, see Resnik (1997).

[42] The relevance attributed to the identity conditions of structures further distinguishes Shapiro's (1997) position from some Resnik's views: «Resnik [1981] seems to hold that there is no such identity relation, arguing that there is no "fact of the matter" as to whether two structures are the same or different, or even whether two systems exemplify the same structure (but see Resnik [1988, 411 note 16]» (Shapiro, 1997, p. 92).

isomorphic relations between them, which posit structures into a one-to-one correspondence preserving the relevant relations:

> We take identity among structures to be primitive, and isomorphism is a congruence among structures. That is, we stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one. (*ibid*.).

The notion of isomorphism, in fact, fits better with *ante rem* structuralism (where structures are axiomatized *directly*) than the notion of structural-equivalence (which identifies structures with equivalent types of *systems*, thus establishing 'sameness' of structures from the systems instantiating them).
It is then worth noting that, in Shapiro's (1997) view, the following principle holds:

(2) $[S] = [S'] \longrightarrow [S] \cong [S']$

This principle is to be distinguished from the abstraction principle (1) ($[S] = [S'] \longleftrightarrow S \cong S'$) presented in section 2.1.1, in which the right-hand side (isomorphism on systems) is re-carved to introduce the left-hand side (identity of structures). By contrast, Shapiro defines an identity relation on *structures themselves*: «because structures themselves are in the ontology, we need an identity relation on structures» (Shapiro, 1997, p. 92).
Moreover, as observed by Shapiro (p. 93), structures are identical if they are isomorphic, but the converse does not hold – in accordance with the conditional expressed in principle (2) – for it is subject to some counter-examples. Shapiro (1997, p. 91) mentions the following case:

> Intuitively, one would like to say that the natural numbers with addition and multiplication exemplify the same structure as the natural numbers with addition, multiplication, and less-than. However, the systems are not isomorphic, for the trivial reason that they have different sets of relations.

2.2.2. Shapiro's conception of objects

With these clarifications at hand, let us now examine the nature of places in abstract structures. Shapiro distinguishes two main ways of thinking of them: firstly, consistently with *in re* structuralism, they can be understood in terms of the objects occupying them, as the so-called 'places-are-offices' perspective shows. According to this interpretation, places just correspond to *offices* or *roles*, which can be played by different sorts of objects or people. In Shapiro's example (p. 10):

> […] we speak of different people who have held the office of Speaker of the House, different people who have played shortstop, and different pieces of wood and plastic that have played the role of white queen's bishop.

In this perspective, the relevant properties and relations concern the objects occupying the positions, and not the positions themselves.

Secondly, mathematical positions can be articulated according to the 'places-are-objects' perspective, in which (empty) places are legitimate objects in themselves, denoted by singular terms and displaying their own properties and relations. According to Shapiro, then, even though mathematics investigates structures in the first place, mathematical singular terms actually refer to places, independently of any objects occupying them, and thus mathematical statements can be read according to their surface grammar.[43] Significantly, Shapiro deems the difference between a place and an object a relative one: though there is an intuitive distinction between places and objects, the notion of object depends on the structure at hand: «what is an office from one perspective is an object – and possibly an officeholder – in another» (Shapiro, p. 10). For example, mathematical structures may include places which are occupied by other structures. Moreover, if places in a structure are proper objects, then these objects occupy the places as understood in the 'places-are-objects' perspective and structures turn out to be systems that exemplify themselves.

---

[43] This holds in particular for *pur*e mathematics, to which Shapiro implicitly restricts his investigation.

2.2.3. The 'places-are-objects' perspective

Places as objects are entities of a special kind, whose essential identity is completely determined by the structure they belong to: according to Shapiro, to say that the number two is the second position in a particular progression (i.e the *ante rem* progression of the numbers) suffices to characterize it as a completely determinate entity:

> Roughly speaking, the essence of a natural number is the relations it has with other natural numbers. There is no more to being the natural number 2 than being the successor of the successor of 0, the predecessor of 3, the first prime, and so on.[44] (Shapiro, 1997, p. 6).

Hence, Shapiro's view is more explicitly committed to the third structuralist thesis (3) outlined by Reck and Price (2001), according to which «mathematical objects have no more than can be expressed in terms of the basic relations of the structures».

In other words, all that matters about mathematical objects are their *structural* properties, whose nature should be examined in more details. Schiemer and Korbmacher (2018) argue that structural properties in *ante rem* structuralism are consistent with both a *definability account* and an *invariance account*. The definability account describes structural properties as the properties which are determined through the primitive relations of a given structure, definable in the language of the relevant mathematical theory. As claimed by Shapiro (2008, p. 286): «define a property to be structural if it can be defined in terms of the relations of a given structure». The invariance account, by contrast, deem structural the properties that can be inferred through a process of abstraction (Dedekind's abstraction): in Linnebo's (2008, p. 64) words,

> […]a structural property can now be characterized as a property that can be arrived at through this process of abstraction, or, equivalently, a property that is shared by every system that instantiates the structure in question.

---

[44] Cf. Benaceraff (1965, p. 70): «to be the number 3 is no more and no less than to be preceded by 2, I, and possibly o, and to be followed by 4,5, and so forth. And to be the number 4 is no more and no less than to be preceded by 3, 2, 1, and possibly o, and to be followed by…».

This means that a property of an object within a system is structural if it remains invariant under structure-transformation and, then, it is shared by the corresponding objects in isomorphic systems.

The reduction of objects to their structural properties has been expressed by Parsons (1990, pp. 334-5; 2008, p. 106) in terms of an *incompleteness claim* about objects, which further clarifies their very peculiar nature. According to Parsons, places in structures are incomplete in two main ways: first, they are incomplete as fictional characters, in the sense that «fictional objects are taken to be undetermined with respect to properties and relations whose holding or not cannot be reasonably inferred from the story» (Parsons, 1990, p, 334). Second, the incompleteness of mathematical objects is even more radical than that of fictional characters, because while some properties of a novel's character can be understood independently of the novel itself,[45] the structural properties of places as objects cannot be defined outside the structure. Parsons clarifies this point as follows:

> There is at least some level of understanding of this kind of simple mathematical notions like addition, multiplication, or set membership, and of more complex ones such as curve or surface or computation. On a purely structuralist view, however, none of these notions is fixed in a way in which the non-fictional vocabulary used to describe a fictional situation is. (Parsons, 1990, p. 335).

I will go back to the definition of Shapiro's 'places-are-objects' perspective in terms of their structural properties in chapter 5, in which I will develop my own account of Weak Mathematical Structuralism (WMS). In this context, I will favour the invariance account of structural properties and show that Shapiro's conception of objects is subject to some counter-examples – concerning non-structural properties of objects – which pave the way to a different conception of objects in non-eliminative structuralism.

In the next section, it will be useful to compare Shapiro’s interpretation of mathematical objects with OSR’s articulation of quantum particles in the philosophy of science, as the two positions raise related problems concerning the attempt of defining objects in purely structural terms.

---

[45] For example, we have some notions of Sherlock Holmes even independently of Conan Doyle's novel: «Sherlock Holmes is a detective in a sense that we can take to be fixed, also when we consider other detectives (real or fictitious). We have, independently of the story, an understanding of notions such as that of a detective, of a murder, of London, of Baker Street(since these are real places)» (Parsons, 1990, p. 335).

2.2.4. Objects in OSR *vs* objects in *ante rem* structuralism: a comparison

Shapiro's places as objects have a number of analogies with objects as understood in OSR. However, as I showed in chapter 1, OSR comes in a variety of flavours, so it is worth to investigate which version of OSR appears comparable with *ante rem* structuralism. Some ontic structuralists point to *Eliminative OSR* as the scientific structuralist counter-part of *ante rem* structuralism (Busch, 2003). However, here I take a different route. Recall that *Eliminative OSR* does not actually have room for objects in the fundamental ontology of the world. By contrast, *ante rem* structuralism is committed to the existence of mathematical objects insofar as they are understood as mere places in the structures they belong to. Indeed, the 'places-are-objects' perspective takes places to be objects in themselves, with their own properties and relations. At the same time, *Moderate OSR* posits objects and structures on the same ontological floor, a strategy that does not plausibly fit well with *ante rem* structuralism – where structures are *prior* to objects and obtain independently of any examplifications. For this reason, I will focus on the comparison between what I called *Priority-based OSR* and *ante rem* structuralism. Both views accepts the existence of objects but reduce them to *nodes* or *positions* in the structures they belong to, claiming that objects are nothing but the relations in which they stand. As a result, they are *secondary* or *derivative* on structures – in a way that will be specified in chapters 4 and 5. In *ante rem* structuralism this idea is expressed by the reference to *structural properties*. *Priority-based OSR* is not as explicit on this, but the appeal to a *contextual identity* for quantum particles clearly suggests that physical objects are identified with their relational properties which – together – constitute the relevant physical structures. On this basis, it is plausible to say that physical objects – exactly as mathematical ones in *ante rem* structuralism – are entirely defined by their structural properties (see chapter 4, sec. 4.2.1, for a more specific understanding of structural properties in OSR). Even though *Priority-based OSR* acknowledges that quantum particles may have intrinsic properties, these properties – the totality of which is *shared* by quantum particles in entanglement structures – are not relevant for their individuation, which thus turns out to be entirely structural. The same holds for *ante rem* structuralism, where numbers are determined for their very identity by the structure. In other words, objects in OSR and objects in *ante rem* structuralism share what Linnebo (2003, p. 97) defines the S*carce Properties Intuition*, captured by Shapiro's (1997, p. 73) claim that «there is no more to the individual numbers "in themselves" than the relations they bear to each other».[46] Translated in the terms of OSR, the *Scarce Properties Intu-*

[46] For a criticism of this thesis in *ante rem* structuralism, see Linnebo (2003, pp. 97-98).

ition results in the following assumption: «these putative objects [*the objects posited by OSR, if they exist*] have no identity or distinguishing features beyond what is conferred by the structure» (French, 2010, p. 98).

The comparison between mathematical structuralism (i.e. *ante rem* structuralism) and OSR can be outlined as follows:

Mathematical structuralism: a number is a place in the number structure and the number structure exists independently of any exemplifying concrete system.

Ontic structural realism: an electron is a node in the electron structure and the electron structure exists independently of any exemplifying concrete system. (ibid.)

It is worth noting, however, that French (2006; 2010) regards the comparison between mathematical and scientific structuralism as misleading:

> The quantum structure, say, does not exist independently of any exemplifying concrete system, as in the *ante rem* case, it *is* the concrete system! But this is not to say that such a structure is simply *in re*, because the ontic structural realist does not—or at least should not—accept that the system, composed of objects and relations, is ontically prior to the structure. Indeed, the central claim of OSR is that it is the structure that is both (ultimately) ontically prior and also concrete. (French, 2006, p. 176).

Still, note that, on *Priority-based OSR,* quantum particles – despite spatio-temporal and causally effective – are treated as *if they were* nodes/positions, for what really matters about them are their structural, state-dependent properties. If we insist on the distinction between physical and mathematical structures (see French and Ladyman, 2003b),[47] nothing prevents us from drawing a *metaphysical* comparison between the properties of physical and mathematical objects, yet keeping in mind that while the former inhabit the physical world, the latter are ascribable to the abstract, mathematical level described by *ante rem* structures. The comparison advanced here just aims at suggesting

[47] For a specific interpretation of the relationship between physical and mathematical structures see chapter 4. (sec. 4.2.2).

that both physical and mathematical objects are reduced to their structural properties, with obvious problems in both frameworks. OSR is subject to the 'relation without *relata* objection', concerning originally the eliminative interpretations of OSR but also troublesome for those non-eliminative conceptions, such as *Priority-based* OSR, positing a very *thin* notion of objects, actually indistinguishable from a 'no-objects at all' conception. In the next section, I will show as *ante rem* structuralism raises related worries. In chapter 4 and 5, I will advocate a unified strategy to avoid these difficulties by articulating a more substantial conception of objects – to be applied in both the scientific and the mathematical domain – where objects are endowed with both structural properties and non-structural properties, sufficient to establish the numerical diversity of objects.

2.3. *Ante rem* structuralism and the identity problem

As scientific OSR, Shapiro's 'places-are-objects' perspective is subject to a number of objections; in particular, three main problems can be distinguished:

1) *the* '*objects problem*': to which extent are positions in a structure legitimate objects in themselves? In fact, places as objects – though more substantial than places as offices – appear to be too structurally defined to avoid resulting in a position where there are no objects (or even acceptable *entities*) at all.[48]

2) *The* '*reference problem*': even if positions, after all, can be admitted as proper objects, how can we effectively refer to them?

3) *The* '*identity problem*': on a structuralist conception of objects, structurally indiscernible objects are to be numerically identified with each other, in contrast with the mathematical practice.

In addition to problems (1)-(3),[49] a quite separate issue is that of *cross-structural identities*, concerning whether, for example, the natural number 2 should be identified with the real number 2. Shapiro (1997, p. 82) holds that while identifying numbers from different structures could be convenient – even wise, occasionally – cross-structural identities are a matter of *decision* and *stipulation*, not a matter of *discovery*. More precisely, there are no cross-structural identities to start with, and such identities are eventually established by decision and convenience. Shapiro (2006, pp. 128-131) advances a different interpretation, according to which places from different structures are distinct but semantically indeterminate. More specific remarks about this will be made in chapter 5, sec. 5.3.3.

In what follows, I will specifically focus on the identity problem, which has raised a wider debate in the structuralist literature. Roughly speaking, the identity problem concerns whether Leibniz's Principle of Identity of Indiscernibles (PII) can be maintained within a structuralist

[48] This worry has been introduced in Russell (1903), Benaceraff (1965) and Kitcher (1983); More recently, Parsons (2008, p. 107) has sustained that «it is possible to have genuine reference to objects if the 'objects' are impoverished in the way in which elements of mathematical structures appear to be».

[49] Explicitly illustrated by Leitgeb (2020, part B, sec.1-3.)

framework. This issue specifically emerges when considering non-rigid structures that allow for non-trivial automorphisms (internal symmetries that are not identity mappings). Such structures are composed by *distinct mathematical objects* that – if interpreted as mere positions, in accordance with Shapiro (1997) – turn out to be *structurally indiscernible*. The simplest case of non-trivial automorphism is a 2-elements unlabelled graph with no edges, in which the two nodes are structurally indiscernible. Complex and relative numbers structures provide other interesting examples.[50] Consider in particular the relative number structure $< \mathbb{Z}, + >$, given by the set of relative numbers (0, 1, 2, ...; –1, –2, ...). In this case, the change of sign of the relevant elements introduces a non-trivial automorphism, for which + 1 is mapped with – 1, + 2 with – 2, etc. If we describe relative numbers in purely structural terms and assume the Principle of Identity of Indiscernibles (PII), there is no way to distinguish between + 1 and – 1: the statement "+ 1 is identical to – 1 " appears to be true, and this similarly obtains in any structure $\sum$ upon which an automorphism G can be defined. However, such conclusion is in contrast with the mathematical practice: either *ante rem* structuralism is incoherent, or it simply does not apply to structures with non-trivial automorphism – which, though, are very important cases in mathematics.

Let us now examine two main formulations of this problem in Burgess (1999) and Keränen (2001).

2.3.1. Burgess's objection

Burgess (1999) provides a first characterization of the identity problem in *ante rem* structuralism. The author observes that – despite Shapiro's attempts of making the idea of *ante rem* structures more precise, clarifying their existential conditions – some mysterious elements remain. These elements still concern the definition of mathematical objects as possessing structural properties only. Significantly, this interpretation is not particularly controversial in the cases Shapiro refers to, i.e. the natural numbers structure. Natural numbers, albeit completely reduced to their structural features, can be *univocally* determined by their structural properties: in fact, each natural number has a structural property – expressed in a first-order language – that individuates it as opposed to all the other numbers in the same structure, e.g. the property of coming first in the natural number order, next to first, next-next to first and so on. Still, things get more complicated when handling other mathematical structures. Burgess specifically discusses the complex numbers structure and

[50] Other examples can be found in graph theory, group theory, geometry, etc.

the Euclidian structure which – along with the relative numbers structure presented in section 2.3 – represent interesting cases of non-trivial automorphisms.

Let us examine the first case: in the complex numbers structure, the relevant equations have two roots, which are the additive inverse of each other. Burgess (1999, pp. 287-288) mentions the following example: «there we have two roots to the equation $z^2 + 1 = 0$, which are additive inverses of each other, so that if we call them *i* and *j* we have $j = -i$ and $i = -j$.» Thus, a non-trivial automorphism switches *j* and *i,* making them indiscernible as far as their structural properties are concerned. Given that Shapiro himself acknowledges that the two objects are mathematically distinct, this introduces an evident inconsistency within *ante rem* structuralism.

The situation is even more problematic in the second case: the Euclidian plane is homogenous, then each two points *p* and *q* in the plane are affected by an internal symmetry – exactly as complex numbers, the points in the Euclidian structures are structurally indistinguishable in spite of being clearly distinguished in Euclidian geometry.

Shapiro (1997) does not directly face these objections, and this, according to Burgess (1999, p. 288), is «the most serious omission of the entire book». Actually, Shapiro takes into account some intuitive examples (1997, p. 125) drawn from team sports; nevertheless, Burgess argues, they are not really helpful, as they refer to games – like baseball – which do not include internal symmetries, whereas they avoid to mention sports – for instance, football – in which these symmetries could be rather found.

2.3.2. Keränen's objection

Keränen (2001) outlines a more extensive understanding of the identity problem in *ante rem* structuralism. The identity problem challenges *ante rem* structuralism as a form of *realism* about structures, where structures are understood as free-standing entities independent of the systems instantiating them. By contrast, *nominalist* structuralist views are arguably immune from this concern, as they reduce the ontology of structures to the ontology of the systems instantiating them. Before getting into the details of Keränen's argument, some clarifications on the labels at hand are needed. As Keränen points out, *ante rem* structuralism can be identified as a form of *realism* about structures, as it is standard in the structuralist literature. The reference to *nominalist* structuralist views is less clear if we adopt the taxonomy illustrated so far, but it seems to point to both eliminative structuralism (such as Hellman's, 1989, modal structuralism) and non-eliminative *in re* structuralism (such as set-theoretic structuralism)[51] – to those accounts that, contrary to *ante rem* structuralism, deny that structures are *sui genesis* entities existing independently of the systems instantiating them.

Keränen begins by setting out the identity conditions that Shapiro's *ante rem* structuralism should fulfill. In fact, *ante rem* structuralism is supposed to deal with *objects*, as the 'places-are-objects' perspective shows, and not with more generic *entities*. Objects, unlike entities, require precise identity conditions:

> Given two singular terms 'a' and '&' that denote objects of a given kind, there is a definite fact as to whether or not the identity statement 'a = 6' is true (think apples). By contrast, given two singular terms 'a' and 'b' that denote entities, there need be no definite fact as to whether or not 'a = b' is true (think waves). Thus, one might call objects 'properly individuated' entities. (Keranen, 2001, p. 313)

In particular, the following criteria are laid down:

1) Given an interpreted language L and two singular terms *a* and *b*, the conditions under which *a* and *b* denote the same object should be specified in L.
2) The account of identity provided in (1) should be applied to all the objects in L, and not only to the objects denoted by the singular terms *a* and *b*.

Given requirements (1) and (2) and two quantifiers $\forall x$ and $\forall y$ ranging over the domain of discourse L, Keränen outlines an identity schema (IS) as follows, which should be completed by filling out the blank.

(IS): $\forall x \forall y(x = y \Leftrightarrow \_\_\_\_\_)$.

Keränen distinguishes two main ways of meeting with conditions (1) and (2) and then completing (IS): the general properties account (relying on properties which can be possessed by more than one

[51] The reference to set-theoretic structuralism as a form of nominalism seems more contentious, but such identification is suggested by several remarks in Keränen (2001).

object in the same structure) and the haecceity-account (i.e. where haecceity defines a single object, distinguishing it from all the other objects in the same structure). An *ante rem* structuralist should adopt the general-properties account, as any reference to a primitive haecceity of mathematical objects would contradict the spirit of mathematical structuralism itself – which interprets places as irreducibly structural and primitively defined by the structure, not by intrinsic properties. However, Keränen aims at showing that no general-properties account is actually available to *ante rem* structuralism – any attempt of defining such account will result in the identity problem, at least when it comes to structures with non-trivial automorphisms, such as $< \mathbb{Z}, + >$.

Let us then briefly reconstruct Keränen's argument (see sec. 3; 4) and examine how an *ante rem* structuralist might proceed in defining identity conditions for places in a structure. First of all, if applied to *ante rem* structuralism, a plausible general-properties account should admit as the relevant properties intra-structural relational properties only;[52] more specifically, «only the properties that can be specified by formulae in one free variable and without individual constants» (Keränen, 2001, p. 317). Assuming this interpretation, any two places in a structure would be identical if they share the totality of their intra-structural relational properties, without using individual constants.

Therefore, given a system S, two quantifiers $\forall x$ and $\forall y$ ranging over the places of a structure **S** of the system S, *ante rem* structuralism would complete the blank of (IS) as follows:

(STR): $\forall x \forall y\ (y = y \Leftrightarrow \forall \varphi(\varphi \in \phi \Rightarrow (\varphi(x) \Leftrightarrow \varphi(y))))$

While the resulting identity account appears quite promising in principle, it clearly conflicts with the mathematical practice when we consider more specific examples, such as the relative numbers structure $< \mathbb{Z}, + >$. In this structure, +1 and –1 turn out to be *co-referential* terms as sharing the totality of their intra-structural relational properties, so that the following statement appears to be true: +1= –1. This equivalence relation leads to two main implications: first, given that in *ante rem* structuralism places in structures are intended to be the referents for singular terms, identifying +1 and –1 entails a radically revisionist approach towards the surface grammar of objects. This

[52] This applies Benaceraff's (1965, p. 70) paradigmatic idea that «mathematical objects have no properties other than those relating them to other "elements" of the same structure».

represents *per se* a problem for *ante rem* structuralism, whose main purpose is to embrace an anti-revisionist semantics for mathematical objects, interpreting mathematical statements at *face value*. Second, *ante rem* structuralism fails to capture a basic mathematical truth (+1 ≠ –1) and ends up in underwriting a mathematical falsehood (+1 = –1). According to Keränen, a broader, and even more serious consequence of this approach is the following:

> The realist account implies that most mathematical theories are inconsistent. For example, since 1 and –1 are co-referential, we have 0 = 1 + (–1) = 1 + 1 = 2. But, 0 and 2 do not have the same intra-systemic relational properties and hence, according to (STR), it is not the case that 0 = 2» (Keränen, p. 317).

This is exactly what the identity problem amounts to; a general-property account for *ante rem* structuralism commits to identity statements which, however, are at odds with the mathematical practice.

Significantly, this is not the case in nominalist structuralism (i.e. *in re*/eliminative structuralism), where structures are exemplified by systems and a non-structural background ontology is already in place; on this view, extra-structural properties can be admitted in the relevant set of properties of the general-properties account, and eventually solve the identity problem raised by +1 and –1 in $< \mathbb{Z}, + >$.

> For example, the set-theoretic structuralism can simply say that the '1-place' of ($\mathbb{Z}$, +) is individuated by the property 'being occupied by the "1-element" in ($\mathbb{Z}$, +)' where ($\mathbb{Z}$, +) is any system exemplifying the structure ($\mathbb{Z}$, +). (p. 320).

That being said, Keränen argues that an *ante rem* structuralist could in principle appeal to two last strategies to escape the identity problem:

(1) to provide a general-properties account which allows treating as distinct two intra-structurally indiscernible places.

(2) To claim that no general-properties account (or, more broadly, that no account of identity) is needed at all in *ante rem* structuralism.

For reasons of space, I cannot here reconstruct these strategies in detail (for a more exhaustive understanding of them, see Keränen, 2001, section 6). However, it is worth noting that in Keränen's analysis, neither of them succeeds – *ante rem* structuralism cannot resist the identity problem, no matter which strategy it opts for.

Therefore, Keränen concludes that the very idea of structures as 'free-standing' objects, independent of the systems instantiating them, is in jeopardy; differently put,

> Benacerraf was right all along: if mathematical entities have no properties besides the ones relating them to the other elements in the same structure, they are not properly individuated objects at all. We can now see why he was right. (Keränen, 2001, p. 329).

So, given the aforementioned difficulties, *ante rem* structuralism – as a form of realism – turns out to be a not viable option, and nominalist structuralist views – which do not apparently meet with the identity problem – should in general be preferred.

2.4. Some responses to the identity problem

Several solutions to the identity problem as raised by Burgess (1999) and Keränen (2001) have been proposed in the literature. In the following sections, I will focus on Ladyman (2005), MacBride (2006) Shapiro (2006a; 2006b; 2008) and Ladyman and Leitgeb (2008) as the most promising strategies. Crucially, the majority of them contrasts Keränen's idea that neither option (1) (to furnish a general-properties account in which indiscernible places are distinct) nor option (2) (to deny that an identity account is required) works for *ante rem* structuralism, attempting to accommodate them to different extents. In particular, some proposals introduce weaker forms of PII to deal with in a structuralist context (Ladyman, 2005), where mathematical objects are *weakly discernible*. On that view, one can state the non-identity of mathematical objects without violating PII (or, at least, violating just the stronger versions of PII, which demand for an *absolute* or *relative* discernibility of objects). Others treat identity as a primitive notion (Ladyman and Leitgeb, 2008; Shapiro, 2006a; 2006b; 2008; along with Ketland, 2011; Menzel, 2018), arguing that the identity and diversity of places in a structure is accounted for by the structure itself. According to this solution, *ante rem* structuralism is not compelled to accept some versions of PII and is consistent with the mathematical practice, which sometimes concedes that indiscernible objects may be distinct.

Nevertheless, as it will emerge in the following discussion, both these lines of reasoning are not immune from further objections, thus leaving room to other possible ways of tackling the identity problem.

2.4.1. A weaker version of the Principle of Identity of Indiscernibles (PII)

Recall that one of the main conclusions of the identity problem is that an entirely structural definition of objects (that involved by the 'places-are-objects' perspective) cannot account for those cases which violate the Principle of Identity of Indiscernibles (i.e. structures with non-trivial automorphisms). This, according to MacBride (2006) introduces a serious dilemma for *ante rem* structuralism: either indiscernible places are identical, or their difference is to be found in primitive haecceities. Both horns of the dilemma are troublesome: the former amounts to bad news for *ante rem* structuralism, which fails to vindicate the widely accepted mathematical truth that such places are in fact distinct mathematical objects. The second horn, instead, is basically an old news for an *ante rem* structuralist, whose view would inevitably collapse into traditional platonism – where objects are defined by intrinsic identities or haecceities, rather than by their structural relations.

However, Ladyman's (2005) purpose is to show that an alternative variant of PII can be accommodated in structures with non-trivial automorphisms, thus introducing a third way to the dilemma. This would recover a version of Keränen's option (2), where the general properties account is meant to include *relational* properties.

Such proposal is based on Quine's (1960, p. 230) distinction between *absolute*, *relative* and *weak discernibility* (see chapter 1, sec. 1.1.3). Ladyman (2005, p. 220) focuses on the automorphism case of the complex numbers +*i* and –*i*: while they cannot be either absolutely or relatively discernible – they can be permuted while leaving the structure unchanged – they are *weakly discernible* in virtue of the symmetrical, but irreflexive relation holding between them, i.e. 'being the additive inverse of'. Since an object cannot bear this relation with itself, it follows that +*i* and –*i* are at least numerically distinct, in spite of being structurally indiscernible. This strategy easily generalizes to other automorphism cases, such as the relative numbers +1 and –1 (which are the additive inverse of each other) and points in the Euclidian space (standing in the relation of 'being a distance *d* apart'). On this view, *ante rem* structuralism is able to distinguish indiscernible places insofar as the weakest form of PII and the notion of weak discernibility are established.

Crucially, the same argument has been advanced by Saunders (2003) in the context of the debate on the individuality of quantum particles in entanglement states (cf. chapter 3, sec. 1.1.3). This, according to Ladyman (2005), helps to corroborate the tenability of weak PII, which fits the individuality of both physical and mathematical objects.

However, such proposal is subject to a serious challenge, formulated by MacBride (2006) and investigated in the next section.

2.4.2. The dilemma re-established

MacBride (2006) observes that the structuralist interpretation of objects proposed by Shapiro (1997) – in which objects are nothing over and above the relations in which they stand – and the related identity problem echo the familiar conception of objects as *bundles of universals*.[53] Not surprisingly, this doctrine traditionally meets with a similar problem, concerning objects which are numerically distinct and yet exhibit the same universals. It is noteworthy that in this context an analogous dilemma emerges: either the bundle theory cannot account for the diversity of objects – and then such objects are indiscernible – or a special category of objects, i.e. bare particulars, should be introduced.

On the basis of this analogy, MacBride aims at evaluating what is questionable in Ladyman's (2005) proposal. In doing so, MacBride takes into account Russell's objection to the bundle theory.[54] According to Russell, symmetric and irreflexive relations (for example, the spatial relation of being a mile apart for Max Black's two spheres) are not able to distinguish structurally indiscernible objects. This is because universals are standardly capable of *repetition*. If we assume the theory of objects as bundles of universals – according to which objects are essentially universals – then objects should be capable of repetition as well. This means that an object could in principle stay in the relation of 'lying at a certain distance' from itself, thus being in two places at once. As a consequence, spatial relations do not ensure that two objects *a* and *b* are distinct just because they are a certain distance apart.

Russell (1911-12) concludes that «the terms of spatial relations cannot be universals or collections of universals, but must be particulars capable of being exactly alike and yet numerically diverse» (118), thus introducing bare particulars as a separate category of objects. MacBride (2006, p. 66) points out that this is because «the obtaining of irreflexive relations cannot constitute but *presupposes* the numerical diversity of the terms they relate».

Such objection can be easily rephrased in the context of mathematical structuralism and the identity problem as addressed by Ladyman (2005). MacBride assumes that mathematical structuralists actually endorse a conception of objects as *bundles of relations.* Similarly to what happens with the interpretation of object as bundles of universals, the fact that two mathematical

---

[53] See Allaire (1963); the argument was firstly endorsed by Moore (1900).

[54] Cf. Russell (1911; 1912).

objects – say, +*i* and –*i* in the complex numbers structure – stand in the symmetrical and irreflexive relation 'being the additive inverse of' does not suffice to establish that they are distinct mathematical entities. In other words, Ladyman's solution conflicts with the structuralist idea that objects are nothing but bundles of relations and the numerical diversity between them remains to be accomplished: «Ladyman assumes without question that irreflexive relations are capable of constituting – without presupposing – the diversity of the (otherwise indiscernible) terms they relate» (MacBride, 2006, p. 67). In fact, Ladyman does not motivate this assumption, and he simply invokes the same argument proposed in the context of the philosophy of science by Saunders (2003). However, this is far from being a justification of the present strategy and indeed shows that scientific structuralists too should further justify the appeal to a weak form of PII.

Along the lines of Russell, MacBride ends up supporting a conception of objects as bare particulars: objects should be constituted independently of the relations between them and even before such relations can obtain. In other words, objects should be *predicatively* constituted. Unless a clearer attempt of understanding them *impredicatively* is provided – specifying how there could be objects, not distinguished in advance, whose numerical diversity depends on their bearing an irreflexive relation to one another – then Ladyman's (2005) solution to the identity problem is subject to a serious objection.[55]

On these grounds, other strategies have been advocated, showing that no account of identity is actually needed to tackle non-trivial automorphism cases – thus opting for the second possibility (2) ruled out by Keränen, i.e. an account of identity is dispensable.

[55] The notions *predicative* and *impredicative* have been introduced by Russell (1908) and refer to the distinction between predicative and impredicative definitions, which played a prominent role in the debate between Russell (1908) and Poincarè (1906) about the nature of logical paradoxes. Informally, definitions are impredicative if they refer to a totality to which the defined entity belongs. They are predicative otherwise. Linnebo (2017) mentions «let π be the ratio between the circumference and diameter of a circle» as a case of predicative definitions, in which π is defined solely with respect to some given circle. By contrast, «let n be the least natural number such that n cannot be written as the sum of at most four cubes» is a case of impredicative definitions, for it generalizes over all natural numbers, n included. Impredicative definitions have been considered controversial and viciously circular by Poincarè, Russell and Weyl, while Gödel (1944) has argued that such definitions are legitimate ones. See Linnebo (2017) for a more extensive analysis.

2.4.3. Renouncing PII

This route has been explored to different extents by Shapiro (2006a; 2006b; 2008) and Ladyman and Leitgeb (2008) who share the basic idea that structurally indiscernible objects in non-rigid structures should be given a primitive notion of identity which, consistently with the mathematical practice, suffices to take them as distinct with no need of committing to a form of PII.

Let us start by evaluating Shapiro's (2006a) contribution to the debate. Shapiro (2006a, p. 134) rejects the idea that a non-trivial resolution of the identity problem is a requirement for *ante rem* structuralism:

> Is the individuation task a reasonable demand on a philosophical or scientific theory? This raises the age old problem of the identity of indiscernibles. I do not see how there can be a non-trivial resolution of the individuation task for *any* view that holds that the usual array of mathematical objects exist. The reason is that there are too many objects and not enough formulas.

Recall Keränen's requirement for individuation, which Shapiro presents as follows:

(IND): $\forall x(x = a \equiv \_\_\_\_\_)$.

In order for this requirement to be satisfied non-trivially, the blank should be filled with a formula that does not have a singular term denoting *a*. The problem is that numbers are uncountable, while formulas are countable. Shapiro takes real analysis as an example: assuming the (IND) individuation task, countable-many reals will be individuated but most of reals will be left un-individuated, since the mathematical language provides us with just countable-many formulas. Differently put, the mathematical language does not have the resources to characterize each object univocally. On this basis, Shapiro claims that asking for a non-trivial resolution of the individuation task is too much to ask to *ante rem* structuralism and that just a trivial individuation is available:

> In any case, someone who opts for realism in ontology and accepts the individuation task must either introduce haecceities or adopt some equally trivial resolution. Presumably, this route is not available to the ante rem structuralist, due to the slogans about how places-in-structures are characterized. (Shapiro, 2006a, p. 135).

As pointed out by Shapiro, taking into account haecceites conflicts with the slogans of *ante rem* structuralism – like «the essence of each mathematical object consists of its relations to other places in the same structure» – put forward in Shapiro (1997). Still, leaving haecceites aside, there is another option available, that is to accept that two distinct mathematical objects can have their essential properties in common and yet be distinct.

> If we do not invoke haecceities, then why should we think that distinct objects always have distinct essences? Why think that two distinct objects cannot have *all* of their essential properties in common? Thus, I do not see how the ante rem structuralist is committed to a crucial premise for the identity of indiscernibles [...] (Shapiro, 2006a, p. 140)

Unfortunately, also this claim appears to contradict some of the earlier commitments by Shapiro's (1997). For this reason, Shapiro engages in analyzing away those claims which seem to commit him to a non-trivial solution of the identity problem. Shapiro specifically refers to the following remark:

> Quine's thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this (Shapiro, 1997, p. 92).

In particular, Shapiro regrets the use of the expression 'criteria for identity' which is misleading in suggesting a non-trivial solution to Keränen's (2001) individuation task. Shapiro (2006a, p. 140) specifies that if 'criteria for identity' stands for *individuation*,[56] then he hereby takes the remark back:

---

[56] Shapiro appears to drop this ambiguity insofar as «[...] the individuation task is not to merely *distinguish* any pair of distinct objects from each other, but to *individuate* each object. As Keränen puts it, the job is to specify for each object *a*, 'the fact of the matter that makes *a* the object it is, distinct from any other object', by 'providing a *unique* characterization thereof '»(Shapiro, 2006a, p. 134).

> What I meant [...] was that if we are to develop a theory of structures, then there must be a determinate identity relation between structures [...] Surely the same goes for places within a given structure [...] When it comes to mathematical objects—places within a given structure—identity must be determinate.

In fact, to say that the identity of places in a structure must be *determinate* does not mean that identity is to be *defined* in a non-trivial way – which is actually not possible in *ante rem* structuralism. By contrast, as clarified in Shapiro (2006b; 2008), identity seems to be *presupposed* in the mathematical practice.

The rejection of Keränen's individuation task goes together with the rejection of the Leibnizean PII: PII is consistent with two interpretations, none of which actually satisfies Keränen's demand for individuation: first, if a *metaphysical* interpretation of the principle is in place, entailing that objects come pre-packaged as metaphysical primitives which can be *univocally characterized* by language, formulas and properties, why should we suppose that we are able to univocally pick objects, including abstract ones? Second, if one adopts a *Quinean* interpretation of PII – in which objects are not given in advance but depend on our conceptual schemas – then a *distinguishability principle* obtains: *a* and *b* should be identified as long as we cannot distinguish them by means of our conceptual resources. Shapiro notes that this principle is definitely weaker than PII, for it does not say that we ought to *uniquely characterize* a pair of objects, but just that we should be able to (numerically) distinguish it.

Keränen (2006) questions that Shapiro's trivialization of the identity problem is an effective one, and argues that further discussion is needed. In fact, even if Shapiro rejects haecceitism and endorses the idea that two distinct objects can share their essential properties, he still ends up embracing a form of haecceitism, which undermines the very motivations for introducing structuralism in the first place. That is because Keränen (2006, p. 156) maintains that there cannot be distinct objects which are essentially indistinguishable: «you must still think that there is *something* about the world that is responsible for the objects being two and not one». Denying that non-essential properties are up to this task, Shapiro has no choice but to admit *primitive* facts to distinguish structurally indiscernible objects. But this amounts to adopt haecceitism, contradicting the very spirit of structuralism: while many systems can instantiate a structure, *ante rem* structuralism claims that the structure itself is *unique*. By contrast, with heacceitism, the problem re-emerges at the level of structures, committing to «*indefinitely many distinct copies of the same* *structure»* (Keränen, 2006, p. 159). In a similar vein, Keränen (p. 154) objects that Shapiro has provided compelling reasons to abandon PII, exactly because, once again, «given any domain of objects, there is *some* fact that metaphysically underwrites the distinctness of any two distinct objects in that domain» and this fact should be accounted for by the properties of objects, consisting with a form of PII.

This leads to Shapiro's (2006b) response; Shapiro first concedes, for the sake of the argument, that Keränen's criterion of individuation for *ante rem* structuralism is a legitimate one. If this is the case, one further option is to embed structures with non-trivial automorphisms in richer, rigid structures which meet the criterion. Specifically, Shapiro suggests to replace cardinal structures with their corresponding ordinal structures – and this seems to work for both the automorphism on the complex numbers structure and the Euclidian plane.

> The complex number structure would be replaced with $\mathbb{R}^2$, with complex addition and multiplication defined in the usual way. In effect, this allows us to use the linear order on the real numbers to distinguish i ($\langle 0,1\rangle$) from $-i$ ($\langle 0,-1\rangle$): since the real number 1 can be distinguished from -1, the pair $\langle 0,1\rangle$ is distinguished from $\langle 0,-1\rangle$. Keränen wonders what rigid structure might be associated with the Euclidean plane. The structure of $\mathbb{R}^2$ works here too, if we include the operations of complex analysis. (Shapiro, 2006b, p. 169).

Still, lifting the concession to Keränen's criterion, Shapiro concludes that a primitive notion of identity should be rather introduced.[57] Shapiro considers as an example set-theory, in which facts about membership work as primitives – and that means that there is no need to provide an *account* of them. Unless set-theorists elaborate an account of membership, thus leaving room for further discussion, «[...]why isn't identity another primitive? Why do we have to give an 'account' of identity?» (Shapiro, 2006b, p. 170). Significantly, a primitive notion of identity is not inconsistent with *ante rem* structuralism: in fact, identity can be considered *among the relations of a given structure*, so that haecceities turn out to be relational properties. On these grounds, haecceitism does not force

[57] That is also because, as specified in Shapiro (2008, p. 295), to embed non-rigid structures in rigid structures undermines the prospects of interpreting mathematical language at *face value*, which was one of the main motivations for adopting *ante rem* structuralism.

one to abandon either the earlier slogans of *ante rem* structuralism (such as «the essence of each mathematical object consists of its relations to other places in the same structure») or – even more seriously – a structuralist approach on the whole. By contrast, understanding the identity of places as primitive shows that *ante rem* structuralism fits well with the mathematical practice.

This last point is further discussed in Shapiro (2008): considering the automorphism on +*i* and –*i* in the complex numbers structure, Shapiro (p. 287) notes that «the fact that it is a theorem of complex analysis that −1 has two distinct square roots seems to be enough to distinguish them, or at least enough to convince us that there are two, and not just one».[58] In other words, it is a *mathematical fact* that indiscernible places in non-rigid structures are distinct, «for the identity relation is presupposed throughout the enterprise of mathematics» (Shapiro, 2008, p. 293). Shapiro calls this the *faithfulness constraint*, i.e. to provide an interpretation that reflects the mathematical methodology as much as possible.

The same strategy is advocated by Ladyman and Leitgeb (2008), who compare mathematical structures with *unlabelled graphs* and focus on the unlabelled graph G' with two nodes and no edges as a simple case of non-trivial automorphism.

G’ ○ ○

In G', the two nodes can be permuted while leaving the graph unchanged: neither node in G can be said to be the node *a* and the node *b* – they are s*tructurally* indiscernible. This is an interesting case of non-trivial automorphism, because there is not a symmetric and irreflexive relation holding between the nodes which can weakly individuate them; so, the graph G' violates even weak forms of PII.[59] However, this is far from being a problem for graph theory for «the fact that G’ consists of precisely two nodes is simply part of what G’ is» (Ladyman and Leitgeb, 2008, p. 392). Once again, this is largely motivated by the way graph theorists use graphs in the mathematical practice: the two nodes in G', though interchangeable, cannot collapse into one another, because they would result in

[58] Specifically, Shapiro proposes to treat the automorphism on +*i* and –*i* in the complex numbers structure by understanding '*i*' as a parameter in natural deduction systems (see Shapiro, 2008, sec. 3).

[59] The edgeless graph G' is contrasted with a graph G having an edge between the two nodes: such graph can be considered as the graph-theoretic counter-part of the automorphism on +1 and –1 in <ℤ, +> and +*i* and –*i* in ℂ. Considering G, Ladyman and Leitgeb acknowledge that the two nodes are discernible in virtue of the symmetric and irreflexive relation holding between them, i.e. *x* is connected to *y* by an edge (in G), consistently with Ladyman (2005). However, of course this solution is not applicable to the edgeless graph G' they take into account, thus showing that a different solution to the identity problem is required.

a different (smaller) graph. On this basis, we are actually justified in defining the graph G' as a graph with exactly *two* nodes and no edges. Along the lines of Shapiro (2006b), Ladyman and Leitgeb (2008) advance the idea that the identity and the non-identity between nodes are considered *among the structural relations* that the nodes bear to each other. Differently put, each node is identified with its structural relations – exactly as *ante rem* structuralism requires – with the only difference that structural relations *include* identity as an irreducible fact.

This example of the graph G' is meant to vindicate two main conclusions:

> (i) the identity or difference of places in a structure is not to be accounted for by anything other than the structure itself, and that
> (ii) mathematical practice provides evidence that this is exactly the way in which mathematicians themselves conceive of places in structures. (Ladyman and Leitgeb, 2008, p. 389).

Still, a number of questions remains open. To begin with, Button (2006) has criticized this approach by raising an *epistemological* and a *metaphysical* objection. First, it is unclear how we can have an epistemic access to primitive identity facts. Second, it is questionable whether indiscernible objects whose distinguishability relies on a primitive notion of identity (such as the two nodes in G') are objects in a proper sense. Ladyman and Leitgeb address both issues by invoking the mathematical practice: we can have access to graphs like G' simply because these graphs exist (and, being isomorphic, G' is unique) and are actually used in graph theory. Similarly, the nodes in G' – and the other indiscernible mathematical objects whose identity is *presupposed* in mathematical methodology – have a legitimate *objecthood* for graph-theorists quantify and apply them precisely as if they were objects.

Another serious criticism has been put forward by Parsons (2008, p. 108). Parsons claims that the solution to the identity problem provided by Ladyman and Leitgeb (2008) (and, apparently, that suggested by Shapiro 2006a; 2006b) brings about a 'dismissive attitude' which is given intuitive force and motivations by the reference to the mathematical practice. Parsons appears to suggest that taking identity as a basic relation of the structure, simply assuming that is a mathematical theorem that $a \neq -a$ in a structure with non-trivial automorphism, leaves largely un-explained *which* is the negative and the positive term and *what it means* to take them as the positive and the negative term respectively – something that *ante rem* structuralism, as a *philosophical* position, is supposed to spell out.[60] On this basis, Parsons acknowledges that the debate is likely to continue. In much the same way, I hold that it is worth to explore alternative solutions to the identity problem. To this aim, in chapter 5, I advance a third way to approach the identity problem which involves neither a primitive notion of identity nor the reference to a weak form of PII. Before doing that, a deeper look at the metaphysical claims involved in both scientific and mathematical structuralism is needed, along with a more precise understanding of those metaphysical notions which capture these claims in contemporary analytic metaphysics.

---

60 For this reason, Parsons (2008, p. 108; see also Parsons, 2004, chapter 4) has proposed his own solution to the identity problem. The core idea is that structures can be distinguished in *basic* and *constructed* structures. Basic structures (such as the natural numbers structure, the real numbers structure and sets) do not come with the obligation of being constructed within other structures. By contrast, constructed structures (such as fields) need to be derived from basic structures, inheriting their main properties. The point is that basic structures should not have non-trivial automorphisms, so that indiscernible objects in structures with non-trivial automorphisms can be distinguished in virtue of the properties of the basic structures they are constructed upon. However, Parsons acknowledges that some structures cannot be so easily constructed from basic structures: a contentious example is the Euclidean plane.

3. A (Non-Foundationalist) Metaphysical Toolkit for Structuralism

So far, scientific and mathematical structuralism have been illustrated, focusing on scientific OSR and mathematical *ante rem* structuralism respectively. The two accounts have lot in common, not only with respect to the notion of objects they outline – and the related objections – but also as concerns the relationship between objects and structures. Typically, objects are *secondary* or *derivative* on structures. The present chapter aims at clarifying this relationship in light of some concepts which enjoy a privileged status in contemporary metaphysics: *fundamentality* and *priority*. For example, Jonathan Schaffer says that «metaphysics is about what is fundamental, and what derives from it» (2009, p. 379). The interest in the fundamental reflects the idea that some entities of our world are its basic building blocks, which *make up* or *build* everything else that exists (Bennett, 2017). Priority is a related concept. It captures the idea that some entities are dependent in some ontological sense on others one. So understood, priority fits well with the view that the reality we inhabit is *hierarchically structured*. While this picture is admittedly vague, it captures the idea that lower-level entities are ontologically prior to higher-level ones. At the bottom of the hierarchy, the fundamental physical entities lie. If, among other things, metaphysics is in the business of elucidating fundamentality and priority, then scientific and mathematical structuralists can benefit from some concepts of the metaphysicians' toolkit to clarify their theses. In the present chapter, such deeper metaphysical perspective will be examined in general. More specific remarks concerning scientific and mathematical structuralism will be provided in chapters 4 and 5 respectively.

Standardly, priority and fundamentality are the basic notions *Metaphysical Foundationalism* (MF) is committed to. This picture has raised a broad discussion on metaphysical dependence – more precisely, on *ontological dependence* and *grounding* as the notions which best capture the idea that reality is structured and hierarchically arranged in different levels. Not surprisingly, the notion of fundamentality has been often expressed in terms of *ontological independence* or *being ungrounded*. Dependence, grounding and their main formulations will be presented in details, showing that despite significant analogies, they are ultimately two different notions. The debate on MF is also deeply entangled with the concept of *metaphysical explanation*, for investigating the structure of reality requires an interpretation of explanation that is distinctively non-causal in character: «talk of metaphysical explanation is bound up with talk about reality's structure» (Thompson, 2019, p. 109). Significantly, both grounding and dependence have a strict link with metaphysical explanation, but just the former has been even *identified* with metaphysical explanation, for reasons which will be reconstructed below.

However, MF is subject to different counter-examples, as recently discussed by Bliss and Priest (2018), which deeply undermine the concepts of fundamentality and priority as traditionally understood; among the alternatives to MF, the most significant positions are Infinitism, which states that there are no foundational elements, and Coherentism, according to which everything depends upon/is grounded in everything else. Although both approaches are explanatorily desirable and metaphysically plausible, they are not immune from objections. On this basis, I will articulate *Weak Structuralism* (WS) as a further alternative to MF, based on a mutual – but not exactly symmetrical – grounding relation. In fact, WS will suggest a natural counter-example to MF, which questions the standard properties of grounding and deserves to be taken seriously along with the other non-foundationalist accounts. WS will emerge as a middle-ground approach which is both logically acceptable and metaphysically favourable.

3.1. Metaphysical Foundationalism (MF)

Metaphysical Foundationalism is the view that reality has a foundation. The most common version of MF is *pluralistic* MF, which assumes a multitude of basic elements as the fundamental constituents of reality.[61] As acknowledged by Tahko (2015), the fundamental level is generally taken to be *smaller* than the upper levels, so that the structure of reality resembles a cone. The idea that there is a scale to the structure of reality is best vindicated by the Standard Model of particle physics, admitting 16 fundamental particles at the bottom and all the other entities in the world at different levels up to the ultimate end of the spectrum. Actually, MF is consistent with three main interpretations of the structure of reality (closed; open at the upper level but closed at the lower level; open at the lower level but closed at the upper level) involving specific differences but still committed to the broadest claim that there is a fundamental level. More specifically, MF claims that reality is hierarchically arranged with chains of entities ordered by anti-symmetric (AS), transitive (T) and anti-reflexive (AR) relations of dependence/grounding terminating in something fundamental – the extendability assumption (everything metaphysically depends upon/is grounded in everything else) is rejected (¬ E).

To put it more schematically, MF is committed to the following theses:

1. *Hierarchy*: reality is hierarchically structured by ontological dependence and grounding relations which are anti-symmetric (AS), anti-reflexive (AR) and transitive (T).
2. *Fundamentality*: there is something fundamental (¬ Extendability).[62]

The hierarchy assumption (1) and the fundamentality assumption (2) will be assumed as the main critical target of the discussion, because they adequately capture the theoretical core of foundationalism and the importance of dependence and grounding in structuring reality.

The focus on these theses also highlights the role of the non-extendability (¬E) conception in MF; in fact, this property delineates a precise link between metaphysical dependence and fundamentality (1) which is significantly challenged in the non-foundationalist perspectives I am going to

[61] Pluralistic MF is opposed to monism, which identifies the fundamental level with a unique element (Shaffer, 2009).

[62] Bliss and Priest (2018) also mention the (3) C*ontingency Thesis* (whatever is fundamental is merely contingently existent) and the (4) *Consistency Thesis* (the dependence structure has consistent structural properties). However, these theses are controversial and also not directly related to the topics here presented. For this reason, I focus on the *Hierarchy* and the *Fundamentality* assumptions, which are required by any version of MF as sufficient and necessary.

take into account. As acknowledged by Bliss and Priest (2018), there are two main ways of making sense of the fundamentality assumption: *being well-founded* and *having a lower bound*. To say that a relation is *well-founded* is to say that «[…] (i) chains ordered by the relation downwardly terminate in a *fundamentalium*, and (ii) that there is a finite number of steps between any member of a chain and the *fundamentalium* that it terminates in.» (Bliss and Priest, 2018, p. 6). By contrast, *having a lower bound* is defined as follows: «where a relation is bounded from below, there need not be a finite number of steps between any member of that set and the *fundamentalium* that grounds it» (*ibid*.). Both interpretations admit a fundamental basis and then are compatible with ¬ E. However, more specific examples of *being well-founded* and *having a lower bound* make the distinction between them more precise. A standard case of *being well-founded* is set-theoretic well-foundedness: «An order < on a domain is said to be well-founded if every non-empty subset of that domain has a minimal element» (Cotnoir and Bacon, 2012, p. 187). To put it in grounding terms, «the grounds of any truth that is grounded will "bottom out" in truths that are ungrounded» (Fine, 2010, p. 100). The idea of well-foundedness clearly rules out the possibility of having an infinite structure which does not itself terminate – the *fundamentale* of a given structure is part of the structure itself. However, some have observed that well-foundedness so understood is too strict, for it leaves aside infinite structures which though are in principle acceptable in MF (Dixon, 2016; Rabin and Rabern, 2016). This is the sense of fundamentality which is captured by *having a lower bound*. A structure can be infinitely descending – thus violating well-foundedness – and yet being founded in an independent element that is not part of the structure at play:

> Consider an infinite chain of dependence $f < ... d_3 < d_2 < d_1$, where the chain of dependent entities d_n terminates in some minimal element *f*. Now, if we take a subset of that chain of dependence without the minimal element *f*, then we are left with a chain that lacks a <-minimal element, hence violating the set-theoretic definition of well-foundedness [...] A lower bound of a set does not need to be an element of the set itself. Consequently, the chain $f < ... d_3 < d_2 < d_1$, is bounded from below. (Tahko, 2018, sec. 2.).

Therefore, *having a lower bound* expresses a distinct and broader sense in which fundamentality can be understood, which is partially consistent with the idea of non-terminating structures. Still, the most common interpretation of MF is intuitively committed to chains of entities which are *fini-*

tely grounded, and then well-founded. This is the interpretation of ¬ E I will mainly contrast in the following discussion, showing that there is room for alternative – and still largely un-explored – conceptions. Let us now explore ontological dependence and grounding as standardly formulated in a foundationalist perspective.

3.1.1. Ontological Dependence

Ontological dependence captures a variety of relations of non-self-sufficiency. Broadly speaking, ontological dependence is a metaphysical and explicative notion which conveys a distinctively non-causal priority relation among *entities*; as standard examples, consider the following claims:

1) «a set ontologically depends upon its members».
2) «electricity ontologically depends upon electrons».
3) «God is ontologically independent» (Tahko, 2015, p. 94).

In particular, an entity is said to be dependent on another entity either for its *existence* or for its *identity* – including its properties as well. Early analyses of dependence proceeded in existential-modal terms, by focusing on the concepts of necessity and possibility. This fits well with the first claim (1), in which a form of *rigid existential dependence* holds: a set cannot exist if its elements do not exist as well. In the second claim (2), a more generic notion of existential dependence is at play: the existence of electricity depends on the existence of (some) electrons. Let us define it *generic existential dependence*.

A more recent trend is that of analyzing dependence in non-modal terms, and specifically in terms of *essence* or *identity.* The third case (3) is an example of *essential independence* – it is part of the essence of God to be ontologically self-sufficient. For a case in which essential dependence holds, consider the relation between an event and its participants: «it is part of the essence of Socrates' death that it exists only if Socrates exists» (Lowe, 2013, p. 195). *Identity dependence* captures a further way of understanding case (1): a set depends for its identity on the identity of its members, in a sense which will be specified above.

Let us then define these varieties of dependence more formally and start with *rigid existential dependence*:

Rigid existential dependence (EDR): *x* depends for its existence upon *y* $=_{df}$ Necessarily, *x* exists only if *y* exists.

(EDR) is rigid because the existence of *x* depends on that very *y* – and not, for example, on entities which are of the kind of *y*. The main problem of this formulation of dependence is that it is too coarse-grained, for it fails to capture a variety of cases: Tahko (2015, p. 96) mentions the dependence of a living organism on its parts. While it is correct to say that an organism depends for its existence on its parts, we know that such parts can change and yet the organism survives: «*which* objects those parts are is inessential – and consequently it is not the case that it depends for its existence, in the sense defined by (EDR), upon any one of those parts» (*ibid*.). The present example is better accounted for by *generic existential dependence*, presented as follows:

Generic existential dependence (EDG): *x* depends for its existence upon *Fs* $=_{df}$ Necessarily, *x* exists only if some *Fs* exist.

In (EDG), the existence of *x* just requires that some *Fs* exist; consider again an organism and its parts. On (EDG), it is clear that the existence of an organism depends for its existence on the existence of proper parts, without any constraints on *which* parts they should be.

Despite being broader than (EDR), (EDG) is still subject to those counter-examples which affect a modal-existential analysis of dependence. The first counter-example is from Tahko and Lowe (2020) and shows that modal-existential dependence is committed to identifying an object with its essential properties, where essential properties are taken to be property instances. Taking into account the relationship between Socrates and his life, the existence of Socrates's life depends on the existence of Socrates, but also the existence of Socrates depends on the existence of Socrates' life. This requires identifying Socrates with his life – Socrates' life is anything different from Socrates himself, for its existence necessarily coincides with his – although several considerations suggest that they cannot be identical, «his life was long, but came to an abrupt ending, while Socrates himself was snub-nosed» (Tahko, 2015, p. 98). Another serious objection has been raised by Fine (1994) with respect to *necessary existents* such as numbers. In particular, the modal-existential analysis of dependence has the unpleasant implication of making everything existentially dependent upon necessary existent objects. Take Socrates and the number 2: necessarily, the number 2 exists if Socrates does, but this does not mean that Socrates depends for its existence on the existence of the number 2 – this would be an evidently wrong conclusion.

On these grounds, alternative and more fine-grained accounts of dependence have been proposed, i.e. *essential dependence* and *identity dependence,* which are developed in non-modal terms. In fact, there are several cases of dependence which intuitively go beyond modal-existential dependence – in both its rigid (EDR) and generic (EDG) version. The third case (3) illustrated above, concerning the essential independence of God, provides a simple example.

Essential dependence has been introduced by Fine (1995) as an attempt of accounting for the idea that «the relevant connection between the existence of *x* and of *y* is not that it be necessary that *x* exist only if *y* does but that it be an essential property of *x* that it exist only if *y* does» (Fine, 1995, p. 272). For this reason, this interpretation of essential dependence is defined *essential (existential) dependence,* and it includes a rigid and a non-rigid version. *Rigid essential (existential) dependence* is defined as follows:

Rigid essential (existential) dependence (EDER): *x* (rigidly) essentially existentially depends upon *y* $=_{df}$ It is part of the essence of *x* that *x* exists only if *y* exists.

As in rigid existential dependence, (EDER) requires that it is part of the essence of *x* that *x* exists only if that very *y* exists.

(EDER) is to be distinguished from *generic essential (existential) dependence*:

Generic essential (existential) dependence (EDEG): *x* (generically) essentially existentially depends upon *y* $=_{df}$ It is part of the essence of *x* that *x* exists only if some *Fs* exists.

A different version of essential dependence is Fine's (1995) *constitutive essential dependence,* based on the view that essence is expressed in the form of *real definitions,* i.e. propositions or collections of propositions in which we can distinguish the defined term and the terms through which it is defined, that in this case function as the terms on which the defined objects depend: «we may take *x* to depend upon *y* if *y* is a constituent of a proposition that is true in virtue of the identity of *x* or, alternatively, if *y* is a constituent of an essential property of *x*» (Fine, 1995, p. 275).

Constitutive essential dependence relies on the notion of *constitutive essence*, to be distinguished from *consequential essence*:

> A property belongs to the *constitutive* essence of an object if it is not had in virtue of being a logical consequence of some more basic essential properties; and a property might be said to belong to the *consequential* essence of an object if it is a logical consequence of properties that belong to the constitutive essence. (Fine, 1995, p. 276)

Therefore, constitutive essential dependence drops not only the modal terminology, but also the reference to the notion of *existence*, which is controversial on its own.[63] Significantly, Fine's *costitutive essential dependence* can be symmetric in the case of simultaneous definitions:

> Let us suppose that it is of the constitutive essence of Jeeves and Wooster that the first be valet to the second. Then under one view, it will be of the consequential essence of Jeeves to be valet to Wooster and also of the consequential essence of Wooster to have Jeeves as valet. Thus any constitutive essence of two objects will 'dissolve' into consequential essences of each [...] . Indeed, it is plausible to suppose, once simultaneous definitions are allowed, that no sequence of definitions will generate a cycle, with x being defined in terms of y, y in terms of z, and so on all the way back to x. (Fine, 1995, pp. 283-284).

Let us now consider *identity dependence* (Lowe, 1989; 1994; 2005; Tahko and Lowe, 2020), which figures as a special case of essential dependence – in fact, on the definitional account of essence (Fine, 1995), «a thing's essence may be said to constitute its identity» (Tahko, 2015, p. 101). Identity dependence is defined as follows:

Identity dependence (ID): x depends for its identity upon $y =_{df}$ There is a two-place predicate 'F' such that it is part of the essence of x that x is related by F to y.

---

[63] Fine (1995, p. 274) refers to the following shortcomings of an existential form of dependence: «in one respect, existence is too weak; for there is more to what an object is than its mere existence. In another respect, existence is too strong; for what an object is, its nature, need not include existence as a part. In the essentialist/existential account, the missing strength is recovered by importing into the connection of dependence between the being of the two objects what properly belongs to the being itself».

It is important to observe that the relevant notion of identity is not that expressed by the sign '=', referring to the logical or numerical identity relation that each object has with itself. By contrast, identity here corresponds to the *individuality* of an object – what it really is, or which kind of object it is: «to say that the identity of *x* depends on the identity of *y* is to say that *which* thing of its kind *y* is metaphysically determines *which* thing of its kind *x* is» (Tahko, 2015, p. 100). In other words, the individuality of an object is metaphysically determined by the individuality of another object, thus suggesting that (ID) is a form of *metaphysical determination*.[64]

Take again the first case (1), where a set depends for its identity on the identity of its members, as a specific example of (ID). This is a consequence of the criterion of identity of sets, that is the Axiom of Extensionality – stating that two sets are identical if they have the same members:

> We can exemplify (ID) by letting *x* be {*z*} and *y* be *z*, in which case we have that {*z*} depends for its identity upon *z*, because there is a two-place predicate – namely 'being a member of the singleton set' (also known as the *unit set* function) – such that it is part of the essence of {*z*} that it is the singleton set of *z*. (Tahko, 2015, p. 101).

As opposed to Fine's *constitutive essential dependence*, (ID) is asymmetrical: while a singleton depends for its identity upon its members, the converse does not hold:

> For example, since it is part of the essence of singleton Socrates that it is the set whose sole member is Socrates, it is not part of the essence of Socrates that he is the sole member of singleton Socrates. (Lowe, 2013, p. 196).

The asymmetry of (ID) is established by the following principle: «if *x* is *not* identical with *y* and *x* depends for its identity upon *y*, then *y* does *not* depend for its identity upon *x*» (Tahko and Lowe, 2020, sec. 4.2). In fact, *x* and *y* cannot depend on each other for their identity, because this would entail that both *x* and *y* lack well-defined identity criteria, thus introducing a form of vicious circularity.

---

[64] Although it is noteworthy that individuality is sometimes considered as stronger than this and related to other criteria, such as identity conditions, location, persistence, etc.

As it will emerge in the following chapters, some of these formulations of dependence play a crucial role in the characterization of both scientific and mathematical structuralism. Let us now investigate grounding and its relation with ontological dependence.

3.1.2. Grounding

The literature on grounding is vast and intricate. A first issue concerning grounding is whether it is simply the same as ontological dependence or rather an ultimately different notion. Both views have been endorsed in the debate. In this context, I take grounding to be a distinct and, for some reasons, even a preferable notion. Before making this case, let us sum up the main formal features of grounding on the orthodox view (Fine, 2001, 2012; Schaffer, 2009; Rosen, 2010), which are deeply reconsidered in non-foundationalist accounts, such as, for instance, those discussed by Bliss and Priest (2018).

On its broadest construal, grounding captures the idea that some things obtain in *virtue of* some other things. Like dependence, grounding is an *explicative* and a *priority* relation. What are the *relata* of the grounding relations? On the *relational* account of grounding, claiming that grounding is a predicate capturing certain relations, the *relata* of grounding are real-world entities, typically *facts*, *propositions*, or *states of affairs*. This view – which is also the most common in the grounding literature – contributes to distinguish grounding from ontological dependence, which obtains between *entities*. By contrast, the *operational* interpretation of grounding – which focuses on the explanatory role that grounding plays in the metaphysical discourse – takes grounding to be expressed by a sentential operator, namely the operator *because* (i.e. *p* because *q*) and endorses a neutral approach concerning the *relata* of the grounding relations – which can be entities on the one hand and facts/propositions/states of affairs on the other hand.[65] In what follows, for the sake of simplicity, I will maintain that both entities and facts can be the *relata* of grounding, provided that the relational and the operational interpretations of grounding are inter-translatable.

A second important distinction concerns *full* and *partial* grounding. Partial grounding is generally defined in terms of full ground: *x* partially grounds *y* just in case there is something else together with *x* such that jointly fully ground *y* (Fine, 2012, Raven, 2015). The analogy with explanation is helpful to illustrate this. As we can distinguish between a full explanation from its contribu-

---

[65] That is because on the operational account (*p* because *q*), *p* and *q* are taken to be sentences with no obvious constraints on the ontological category these sentences denote.

ting parts, we can separate between full and partial grounds, which are akin to the contributory parts of a full explanation. For instance, take the conjunction A&B. Each of A and B is a partial ground of A&B and each of A and B contributes to a complete explanation of A&B. But typically, neither A nor B on its own suffices to fully ground or fully explain A&B (Fine, 2012, p. 50).

A third distinction is that between *immediate* and *mediate* ground, which is helpful to identify a hierarchy in the grounding relations; immediate grounds are those that do not need to be *mediated* to ground other entities/facts. For example, the immediate grounds of the conjunction P&Q are P and Q. But if we take (P&Q) & R, we need first to identify the immediate grounds of (P & Q) and R respectively. As a result, P, Q and R are *mediate grounds* of the conjunction (P & Q) & R considered altogether. Moreover, grounding can be *strict* or *weak.* As acknowledged by Tahko (2015, p. 107), this distinction is more troublesome, but an intuitive understanding is the following: while strict grounds belong to an explanatory level that is somehow more fundamental than that of the grounded entities/facts, weak grounds occur at the same level of the grounded entities/facts. This raises the worry that «some facts could even be weak grounds for themselves!»[66] (*ibid.*), in contrast with the view that grounding is anti-reflexive, as it is specified below.

With these tools, grounding can be introduced as a *strict partial order*,[67] thus capturing its main formal features: *anti-symmetry* (if *x* grounds *y*, *y* does not grounds *x*), *anti-reflexivity* (nothing grounds itself) and *transitivity* (if *x* grounds *y* and *y* grounds *z*, then *x* grounds *z*). In a standard understanding of grounding, *well-foundedness* is also assumed, with chains of grounding terminating in an ungrounded foundation. Significantly, the same properties have been associated with metaphysical explanation; for example, canonically grounding cannot be symmetric, because this means having not only entities or facts *grounding* each other, but also entities or facts *explaining* each other, contradicting the standard idea that genuine explanations should be anti-symmetric. Each of these grounding features has been challenged, as it will emerged in sections 3.2. and 3.2.1.

Lastly, a further distinction – which is not always explicitly addressed in the grounding literature – concerns two kinds of grounding claims: we can separate between claims of the form '*x*'s existence is grounded in that of *y*' (Fine 2012) from claims of the form '*x*'s identity is grounded in that of *y*'–

---

66 As an example of a weak ground, Tahko (2015, p. 107) considers the following: «Jack's being Jill's sibling, which explains Jill's being Jack's sibling, and also the fact that Jack and Jill are a pair of siblings. Here Jack's being Jill's sibling and Jill's being Jack's sibling occur at the same level of explanatory hierarchy».

67 The definition is from Raven (2013).

developed in strict analogy with the essentialist account of dependence (Raven, 2015).[68] As I am going to show in chapters 4. and 5., this distinction has interesting applications when it comes to the formulation of (scientific and mathematical) structuralist views in grounding terms.

Let us now consider the relationship between grounding and dependence. As anticipated, even though grounding and dependence are intimately related, it is plausible to take grounding as a separate notion. First of all, grounding has a stronger connection with the concept of *metaphysical explanation:* if *x* grounds *y*, then *x* metaphysically explains (or helps explaining) *y*. Grounding explanations are metaphysical in the sense that they involve something that determines non-causally and synchronically the identity and/or the existence of something else. In this respect, Fine (2012, p. 37) associates grounding with «a distinctive kind of metaphysical explanation, in which explanans and explanandum are connected [...] through some constitutive form of determination». Ontological dependence, like grounding, has a clear explanatory role. Still,

> [...]the link to explanation is weaker: even though the existence of water depends on the existence of hydrogen and oxygen, it does not seem to be the case that the existence of hydrogen and oxygen explains the existence of water (Tahko, 2015, p. 104).

In this specific case, what needs to be explained is the ability of hydrogen and oxygen to form water molecules, something that ontological dependence leaves actually un-explained. In the next section, I will more sharply define the relation between grounding and metaphysical explanation, which is crucial for the present discussion. What is important to observe here is that there are also other reasons to keep grounding and dependence apart. In fact, grounding has a stricter link with the notions of priority and fundamentality as well: while it is the case that grounds are *prior to* and *more fundamental than* the grounded facts/entities, at least some interpretations of dependence – the modal ones – admit dependence in absence of priority. Consider for instance *rigid existential dependence*, where saying that *x* ontologically necessitates *y* does not commit to assume *x* as ontologically prior to *y*.[69]

---

[68] Raven (2015, p.14) claims that «when a relation of ground obtains then it does so in virtue of the *natures* of the (constituents of the) grounds, the grounded, or both». For a systematic account of the relation between grounding and identity, see Correia (2017) and Correia and Skiles (2019).

[69]As acknowledged by Tahko (p. 105), it would be quite odd to say that parents, whose existence ontologically necessitates the existence of their children, are ontologically prior to their children. However, it is noteworthy that essential dependence and identity dependence – exactly as grounding – surely express priority, as clarified by Koslicki (2012).

Lastly, the formal features of grounding suggest that grounding is stricter than dependence: in fact, grounding – as a strict partial order – cannot be reflexive. By contrast, reflexivity can be in principle accepted in ontological dependence, claiming that something ontologically depends on itself.

It is worth mentioning that grounding has also interesting connections with causation, modality, truth-making and reduction. For an extensive analysis of these notions, which are not directly relevant for the present discussion, see Tahko (2015, sec 5.5.). However, a quick note on reduction and grounding is needed, for it has implications on the articulation of scientific structuralism in grounding terms (see chapters 4. and 5.): as pointed out by Tahko (p. 113) «although many of the examples might tempt one to think that the grounded entity reduces to whatever it is that grounds it, this is not how grounding is typically understood». This is true if reduction is understood as an *identity* relation between grounds and what is grounded. In fact, on this view, the anti-symmetry, transitivity and anti-reflexivity of grounding appear to be violated. The idea that grounded entities/facts do not reduce to the entities/facts grounding them has been also supported by Audi (2012, pp. 101-102) «grounded facts and ungrounded facts are equally real, and grounded facts are an "addition of being" over and above the facts in which they are grounded». This runs counter the interpretation of the grounded entities/facts as an *ontological free lunch* (Cameron, 2008; Schaffer, 2009; Sider, 2012), i.e. as entities/facts which do not come with ontological costs and do not affect the ontological commitments of a theory, considered with respect to the fundamental/grounding entities only. However, reduction has been differently interpreted and the question whether grounding is a reductive notion actually depends on the conception of reduction adopted.[70]

[70] For example, an understanding of reduction in terms of essence has resulted in the *grounding-reduction* link (Rosen, 2010): if *q* reduces to *p*, then *p* grounds *q*.

3.1.3. Grounding and Metaphysical Explanation

Grounding theorists disagree on the tightness of the link between grounding and metaphysical explanation. Some say that grounding *is* metaphysical explanation (Dasgupta 2014, Raven 2015, Thompson, 2018), endorsing the so-called *unionist* approach. Others opt for a weaker tie (Audi, 2012; Koslicki, 2012; Maurin, 2018; Shaffer, 2012; Trogdon, 2012), claiming that grounding and metaphysical explanation are just closely related. This is the *separatist* approach.[71]

Before presenting the unionist and the separatist approaches, what metaphysical explanation is should be cashed out. As introduced in section 3.1.2., metaphysical explanations involve a non-causal determination relation between entities or facts: *x* metaphysically explains *y* if *x* non-causally and synchronically determines *y*. However, something more should be said in order to identify *metaphysical* explanations as opposed to other kinds of non-causal explanations (for instance, mathematical explanations). According to Thompson (2019, p. 104), metaphysical explanations have a distinctive epistemic import and, as such, they are *answers to questions* – specifically, answers to *what-makes-it-the-case* questions. Answers of this sort should be distinguished from answers to *why* questions, which are provided by different kinds of explanations (which include, among others, casual explanations).[72] Of course, metaphysical explanation is a broad and complex notion, but Thompson's (2019) account suffices for the present purposes and offers a useful starting point to evaluate the separatist and the unionist approaches.

Let us start with the separatist approach, taking grounding and explanation to be distinct notions. The question concerning the relationship between them naturally arises. Typically, on this view, grounding *backs* metaphysical explanation and explanation *tracks* grounding. This perspective relies on the analogy with causation and causal explanations, developed by Schaffer (2012, p. 124):

> Grounding is something like metaphysical causation. Roughly speaking, just as causation links the world across time, grounding links the world across levels. Grounding connects the more fundamental to the less fundamental, and thereby backs a certain form of explanation.

---

[71] The labels *unionism* and *separatism* are from Raven (2015).

[72] To understand the distinction between explanations as answers to *what-makes-it-the-case* questions and as answer to *why* questions, consider the following example provided by Thompson (2019, p. 104): «Asking what makes it the case that the mug is broken demands an answer that has to do with its parts being disconnected; that is what the mug's being broken consists in. Asking why the mug is broken solicits a different kind of explanation (such as that the kitten knocked it off the desk)».

In other words, as causal relations in the world are distinguished from the causal explanations tracking them, grounding relations – capturing the hierarchical structure of reality – just back metaphysical explanations, without being identical to them.

By contrast, the unionist approach identifies the two notions. Grounding is explanatory by its very nature. Therefore, grounding does not simply *share* with metaphysical explanation anti-symmetry, anti-reflexivity and transitivity; grounding itself *just is* metaphysical explanation. Since true grounding claims are generally considered objective, on this account the same goes for metaphysical explanation – which then lacks some of its typical epistemic features, such as context-sensitivity and dependence on subjective understanding. A pointed out by Maurin (2018, p. 1591), this is problematic, for «then the kind of explanation grounding *is* turns out to be radically different from explanation in the 'normal' mind-involving sense».

An argument for endorsing the unionist view is the following: were grounding a relation that just backs metaphysical explanation, we would need a more precise account of the relationship between the two. However, grounding and explanation cannot be related by grounding itself, because metaphysical explanation is generally assumed to clarify grounding, and not *viceversa*, as pointed out by Thompson (2018, p. 114):

> If an appeal to explanation is to shed light on the notion of ground, part of what must be understood is how ground and explanation are related. Here we are told that the relationship between ground and explanation is in fact one of ground, but ground was what we were seeking elucidation of in the first place!

In this context, I am not going to privilege one view over the other, since both the separatist and the unionist approaches are controversial. More modestly, I refer to some principles (as explicitly formulated by Maurin, 2018, p. 1576) which – though generally associated with unionism – suffice to establish the link between grounding and metaphysical explanation without necessarily committing to their identity:

Inheritance: grounding 'inherits' (some of) its properties from explanation (among others, see Raven, 2013).

Involvement: (metaphysical) explanation is such that grounding plays a – possibly indispensable – role in it. (Audi, 2012).[73]

These principles – no matter if grounding *is* or just *backs* metaphysical explanation – give us compelling reasons to take grounding as more suitable than dependence in order to provide a deep insight into the metaphysical structure of reality. More specific benefits of grounding will be illustrated in chapters 4 and 5. For the time being, in the following sections I will focus on grounding and how it can be reconsidered within a non-foundationalist perspective. As I am going to show, since grounding and metaphysical explanation are (at least) intimately related, a broader, non-foundationalist conception of explanation is also suggested.

3.2. Metaphysical Foundationalism reconsidered: non-foundational accounts

With these clarifications at hand, I will now examine Bliss and Priest's (2018) taxonomy, in which different alternatives to Metaphysical Foundationalism (MF) are illustrated. These accounts stem from different combinations of the main foundationalist properties (AS, T, AR and ¬ E) and their negation. The authors outline sixteen possibilities, ten of which – positions (1); (2); (3); (4); (7); (8); (13); (14); (15); (16) – are considered logically acceptable.

Tab.1.

[73] It is important to observe that *Inheritance* and *Involvement* are linked with other principles (see Maurin, 2018, p. 1577). In particular, *Inheritance* is often taken to entail *Informativeness* (that grounding is like explanation informs us about the nature of grounding) while *Involvement* is associated with *Justification* (the fact that there are metaphysical explanations justifies grounding). Still, in this context, I refer to *Inheritance* and *Involvement* – which are more universally accepted among grounding theorists – as the minimal requirements to account for the relation between grounding and explanation.

	AS	AR	T	E	Comments	Special Cases
1	✓	✓	✓	✓	Infinite Partial Order	I
2	✓	✓	✓	×	Partial Order	A, F, G
3	✓	✓	×	✓	Loops	I
4	✓	✓	×	×	Loops	F, G
5	✓	×	✓	✓	x	
6	✓	×	✓	×	x	
7	✓	×	×	✓	Loops Length >0	I
8	✓	×	×	×	Loops Length >0	F, G
9	×	✓	✓	✓	x	
10	×	✓	✓	×	x	
11	×	✓	×	✓	x	
12	×	✓	×	×	x	
13	×	×	✓	✓	Pre-order	C, I
14	×	×	×	✓	Pre-order	C. F, F’, G
15	×	×	×	✓	Loops of any length	I
16	×	×	×	×	Loops of any length	F, F’, G

Among these ten logically possible options, the following special cases can be individuated:

Atomism (A): nothing is grounded in something fundamental.
Foundationalism (F): everything is a fundamental element or is ultimately grounded in something fundamental.
Foundationalism (F’): only the fundamental elements are grounded in themselves.
Foundationalism (G): the fundamental element is unique.
Infinitism (I): there are no fundamental elements.
Coherentism (C): everything is grounded in everything else.

According to Bliss and Priest, these cases (A, F, F', G, I, C) are not only logically, but also *metaphysically possible*: in fact, one may argue that metaphysical foundationalists do not provide compelling reasons to demonstrate that MF is the correct (and the unique) description of reality. The commitment to AS, T, AR and ¬ E is based on the appeal to *intuitions*, as MF is certainly close to the common sense and the search for a ground – in order to avoid an explanatory regress – intuitively terminates in something fundamental. However, as noted by Morganti (2018, p. 258), «the prevalence of foundationalism may well be due *exclusively* to the intrinsic limitations of our cognitive faculties». By contrast, a meta-induction argument seems to support non-foundationalism: some past hypotheses concerning the fundamental level have been now abandoned, suggesting the possibility that there is no such fundamental level. Of course, one could argue that the foundation of reality is a yet to be discovered (set of) element(s). Still, this at least shows that the intuitions foundationalists appeal to do not effectively replace proper arguments. Moreover, they mainly refer to *causal* explanations, which are standardly anti-symmetric and well-founded.[74] As far as *metaphysical* explanations are concerned, the relevant non-foundationalist alternatives are actually advantageous in terms of explanatory power: if compared with MF, they clearly supply additional explanatory resources, which capture further aspects of reality and favour a broader notion of explanation. In this respect, Thompson (2018, p. 116) argues that brute or fundamental facts come with a theoretical cost, because they are something that the theory leaves un-explained: «other things being equal, a theory that carries a commitment to fewer brute facts is to be considered superior because it has more explanatory power—it leaves fewer things unexplained». By contrast, non-foundationalist views deny that these brute facts exist at all: everything is simply explained (at least partially) by everything else, consistently with a *holistic* conception of metaphysical explanation.

Let us now present Infinitism and Coherentism, which prove particularly useful for the present purposes in so far as they more radically question the standard properties of grounding.

3.2.1. Infinitism and Coherentism

Both Infinitism and Coherentism are understood as alternatives to Metaphysical Foundationalism and they are the metaphysical counter-parts of analogous epistemic positions. Epistemic Infinitism introduces an anti-symmetric, transitive and anti-reflexive chain of believes without a foundation. Epistemic coherentism, by contrast, posits a Quinean, highly integrated web of believes. In both cases, justification *emerges* from the overall picture. Let us now present Infinitism and Coherentism as metaphysical approaches.

[74] This highlights the idea that metaphysical explanation constitutes an explanation of a special sort, distinguished from causal explanation (where, for instance, is not the case that something is self-caused): in Bliss and Priest's (2018, p. 12) terms, «what goes for causation here does not (necessarily) go for metaphysical dependence».

Infinitism corresponds to the combination of anti-symmetry (AS), transitivity (T), anti-reflexivity (AR) and extendability (E). This position preserves the idea of a strict partial order, and this justifies the commitment to AS, T and AR. Nevertheless, the idea that grounding relies on a fundamental, ungrounded basis is dropped, so that the extendability assumption (E) can be accepted, as opposed to Metaphysical Foundationalism. Hence, according to Infinitism, reality has a hierarchical structure, but its grounding relations do not terminate in something fundamental – the structure of reality can be open-ended either in the direction in which the grounding relations go or in both directions. Therefore, Infinitism violates the first sense of fundamentality – *being well-founded* – specified above, for it clearly cannot be *finitely grounded.* One could though question wether a version of *having a lower bound* – that is the second, broader sense in which fundamentality can be understood – holds in an infinitist model; in principle, infinititely descending grounding chains are compatible with the idea of a lower bound, that is an independent fundamental element in which the elements of the chain are ultimately grounded. According to Tahko (2018) some more extreme interpretations of Infinitism fail to respect even this condition, i.e. those assuming an *infinite complexity,* with non-fundamental entities being grounded in further non-fundamental entities in the lower level all the way down, without a termination and then without a *full* foundation. Another possibility is that of an infinite repetition, in which the same structure infinitely repeats itself (the *fundamentalia* being consequently identifiable at least at the level of general types of entities). Such structure is defined by Schaffer (2013, p. 505) a «boring infinite descent».

Infinitism is open to criticisms because an infinite regress is apparently introduced: in an infinitist framework, «reality cannot have an infinitist structure because, if it did, everything would simply fail to 'obtain' either its existence and/or the distinctive features that make it the existent that it is» (Morganti, 2018, p. 260). Still, as explained by Morganti (*ibid.*), a regress is not necessarily vicious: in analogy with epistemic infinitism, an infinitist structure of reality is in principle consistent with an emergentist model of being, in which both the existence and the properties of entities are not transmitted from one level to another, but they simply emerge as a whole from the overall structure.[75]

In contrast to Infinitism, Coherentism (¬ AS, T, ¬ AR and E) abandons the very idea of a hierarchical, pyramidal structure of reality and – contradicting AS – posits entities which are symmetrically dependent on each other, so as to obtain *cycles* or *loops*. Since each loop leads back to the starting point, the AR assumption is violated as well: this motivates a serious objection to Cohe-

[75] Morganti mentions hierarchical cosmological models of the universe as specific examples of infinitist structures.

rentism, according to which genuinely informative metaphysical explanations presuppose anti-reflexive relations. Not surprisingly, grounding itself is generally taken to be anti-reflexive (see section 3.1.2). Morganti illustrates some possible strategies to avoid this objection: first, the anti-reflexvity of grounding can be questioned without ending up with logically impossible intepretations of reality, as Bliss and Priest's (2018) taxonomy shows. Second, weaker interpretations of anti-reflexivity can be endorsed in a coherentist model, such as *quasi-reflexivity*: «something is related to (in our case, partially grounds) itself if and only if it is related to something other than itself» (Morganti, 2018, p. 264). Of course, weakening anti-reflexivity in this way commits one to abandon anti-symmetry as well, thus renouncing the standard principle that *a* cannot ground *a*. However, as pointed out by Morganti (2018), the cost of dropping AS comes with significant explanatory advantages and then it is worth to take Coherentism as a relevant philosophical option.[76]

Coherentism plays a crucial role in the present discussion, because it offers a novel non-foundationalist framework to interpret various structuralist positions. In particular, Morganti (2018) considers scientific Ontic Structural Realism (OSR) – and its main varieties explored in chapter 1 – as an interesting case-study in favour of metaphysical Coherentism. OSR typically claims that objects – if they exist – are derivative on the structure they belong to, which is supposed to belong to an upper, more fundamental level. Still, OSR can be also understood in a coherentist sense, with objects being determined by mutual dependence relations which occur at the same, 'horizontal' level.

In the next section, I will outline my own account of Weak Structuralism (WS) as a more moderate variety of Coherentism. At this stage, I will present WS as a broad methodological approach, which needs to be more specifically applied to be fully understood. In this context, my aim is just to state whether this view can be considered as a further account that combines the explanatory advantages of Coherentism with some intuitions of Metaphysical Foundationalism. If this is the case, then WS can be put to work in both scientific and mathematical structuralism, where it may have interesting implications concerning the relationship between objects and structures and the interpretation of objects.

---

[76] For another defense of Coherentism, see Bliss (2014), claiming that the potential infinite regress generated by Coherentism is not necessarily vicious.

3.2.2. Introducing Weak Structuralism (WS)

Weak Structuralism (WS) appears as a specific application of two of the three perspectives (i.e. Foundationalism, Infinitism and Coherentism) illustrated above. In particular, WS can be presented as a mixture of Coherentism and Foundationalism, whereby some (although not all) features of each view are fruitfully maintained and combined. In fact, WS involves a *mutual* (not exactly symmetrical) grounding relation and, under a peculiar interpretation to be explored below, the non-extendability assumption; for reasons of terminological coherence, the latter will henceforth be referred to as the *fundamentality* (F) assumption.

While Coherentism adequately accounts for the (symmetrical) interdependence between objects of the same structure, WS addresses the relation between objects and the structure they belong to. In other words, objects are not only related to each other by symmetrical grounding relations – as in Coherentism – but they also stand in a mutual grounding relation with the structure they belong to. This relation includes an *upward* direction (from objects to structures) involving the *identity* dimension and a *downward* direction (from structure to objects) in which the *existential* dimension is at play. Following Lowe (2005) and Tahko and Lowe (2020), I understand identity as what makes objects the very objects they are. The bi-directionality of the relation relies on the distinction between existential and identity grounding claims (introduced in section 3.1.2.) and can be characterized by two grounding claims holding at the same time, which I shall define *Object Identity* and *Object Existence*:

1) *Objects Identity*: objects are fully grounded for their identity (not for their existence) in the identity of the structure they belong to.

2) *Structure Existence*: structures are fully grounded for their existence (not for their identity) in the existence of the objects constituting them.

For the sake of simplicity, these claims are formulated in terms of *entities*, but note that they can be also rephrased in terms of facts. Each claim is asymmetrical on its own (AS) and does not lead back to the starting point, so that there is not something that has to be self-grounded (AR). However, WS is defined by the mutuality (M) of the global picture, for both claims are required to make WS coherent: objects cannot be *individuated* without structures, but structures themselves cannot *exist* without the *relata* constituting them. Of course, the present account requires introducing a distinction between a s*ymmetric* and a *mutual* conception of grounding.

a) *Symmetric grounding*: if *x* grounds *y*, then *y* grounds *x*.
b) *Mutual grounding*: if *x* grounds$_{(I)}$ *y*, then *y* grounds$_{(E)}$ *x*.

In *Mutual Grounding*, take *x* to be structures and *y* to be objects. The subscripts (I) and (E) denote relativization to existence and identity respectively: if structures ground objects for their identity, then objects ground structures for their existence.
A third grounding claim defines identity criteria for structures independently of objects themselves.

3) *Structure Identity*: the identity of structures is fully grounded independently of objects themselves, with reference to higher, more abstract structures.

Consistently with the metaphysical picture suggested by Metaphysical Foundationalism, *Structure Identity* suggests that there is actually a more fundamental level – that of high-order structures – ultimately grounding objects and structures, which instead are on the same ontological floor. A more precise characterization of claims (1-3) should be provided, clarifying first of all what objects, structures and 'higher' structures exactly amount to. However, recall that in this chapter my intention is just to outline WS as a promising conceptual framework which can be possibly implemented in scientific and mathematical structuralism – where *Object Identity*, *Structure Existence* and *Structure Identity* will be spelled out with respect to the relevant context.

For now, let us lay out some preliminary considerations which will be useful in the following chapters. In scientific structuralism, objects correspond first and foremost to quantum particles and structures to the physical relations in which they stand, such as quantum entanglement. In mathematical structuralism, objects are numbers (e.g natural numbers) and structures the abstract mathematical structures they are embedded in (e.g. the natural numbers structure). In both domains, the problem of providing identity criteria for physical and mathematical structures naturally arises. In chapter 1, I showed that ontic structuralists describe physical structures in terms of symmetry groups of group-theory, which are the mathematical counter-part of quantum entanglement states. According to this interpretation, which historically traces back to Cassirer, Born, Weyl and Eddington, quantum systems are reducible to the relations of symmetry groups. In mathematical *ante rem* structuralism, Shapiro (1997) individuates structures by referring to their isomorphism types, as illustrated in chapter 2. Therefore, both scientific and mathematical structuralism identify abstract, 'higher' structures (symmetry groups and isomorphism types respectively) which are useful to individuate the physical and mathematical structures at hand.

Once I have specified what the entities in *Objects Identity* (1), *Structure Existence* (2) and *Structure Identity* (3) could be, it is worth noting that WS has two major outcomes, resulting from the combination of claims (1) - (3):

i) *Object Identity* (1) and *Structure Existence* (2) entail a peculiar interpretation of objects, whose identity is determined by the relevant structures, but whose existence is necessary to posit relations (and hence structures) themselves. On this view, objects cannot be *individuals* in a proper sense (they are entirely defined by the structure and then lack intrinsic properties) but rather they are more generic *entities* or *things* which are subject to the predication of properties and may exist independently of structures. I label such entities *quasi-thin objects*, endowed with both structural properties and non-structural *kind* properties, i.e. the properties establishing the set of kinds in which physical and mathematical objects come sorted. On the one hand, physical kind properties qualify quantum particles as electrons, muons, etc.; on the other hand, mathematical kind properties qualify numbers as naturals, relatives, reals, etc. I will develop this conception in both scientific and mathematical structuralism, interpreting quantum particles as quasi-thin physical objects and numbers as quasi-thin mathematical objects. The focus on kind properties and their specific features allows us to explain what makes physical and mathematical objects *quasi-thin*: in particular, kind properties provide us with tools to consider such objects numerically distinguishable independently of structures. As I am going to show in chapter 4, sec. 4.2.2, the concept of numerical distinguishability should be kept distinct from the thicker notion of identity – which, as showed by Object Identity (1), is entirely structural. More specific definitions of quasi-thin objects, kind properties and their advantages will be provided in chapters 4 and 5. Some clarifications concerning how WS differs from other positions in scientific and mathematical structuralism will also be given in due corse.

ii) *Structure Identity* (3) clarifies WS's conception of fundamentality (F). Although WS includes two different grounding relations, their mutuality within the overall picture ensures that objects and structures belong to the same fundamental level; hence, the resulting grounding relation appears to be not well-founded in the standard sense (i.e. finitely grounded). Still, the individuation of structures *via* higher structures (symmetry groups for physical structures and isomorphism types for mathematical structures) provides a *bound from below* for both *Object Identity* and *Structure Existence*: given that objects are grounded for their identity in structures, both objects and structures are bounded from below for their identity in higher structures, which are not part of the grounding chain obtaining between objects and structures.[77] The idea of a lower bound captures adequately WS's conception of fundamentality, which clearly differs from that presupposed by Infinitism and Coherentism (assuming forms of extendability) but at the same time is weaker than the form of non-extendability adopted in Metaphysical Foundationalism, where entities are well-founded or finitely grounded.

These clarifications make it plausible to say that WS involves mutuality (M), anti-reflexivity (AR), transitivity (T), and weak-fundamentality (W-F) – understood as being bounded from below. If mutuality is interpreted in accordance with principle (b) above (i.e. according to *Mutual Grounding*), thus including two distinct interpretations of the grounding relation, then it seems compatible with AR. Therefore, WS also advocates a possible strategy to preserve anti-reflexive relations in a context of mutual dependence, distinguished from that proposed by Morganti (2018) for Coherentism. This particular combination of properties distinguishes my own proposal from the other non-foundationalist options considered by Bliss and Priest, for it endorses both a coherentist ingredient with respect to the relationship between objects and structures, and a foundationalist ingredient with respect to the relationship between structures and high-order structures. My intention is to argue that WS introduces a special case of grounding which can be taken seriously along with the other non-foundationalist approaches. First – if M and AR are consistent – WS can be accepted as logically possible.

Second, WS appears to be worth endorsing not only as a *logically* plausible option, but also as an *explanatorily* promising position. On the one hand, similarly to Infinitism and Coherentism, WS entails a broader, non-foundationalist approach to metaphysical explanation; in particular, WS advances a *holistic* explanation, tracking the relation of *Mutual Grounding*, where objects and struc-

[77] Of course, the grounding question arises also for higher-order structures; in this respect, several options may be open; their identity can be either grounded in even higher structures, or assumed as primitive. However, I will not go into detail as regards the issue of the identity of high-order structures, although I acknowledge that this is an open question that is worth to explore.

tures are on a par but structures ground/explain objects for their identity and objects ground/explain structures for their existence. This interpretation seems to be explanatorily advantageous, by providing a wider range of hypotheses in the description of the physical reality. On the other hand, WS does not seem affected by the typical vicious-regress objections that generally concern non-foundationalist positions, thus being also *metaphysically* advantageous:[78] the reference to a mutual, but not properly symmetric grounding relation and to the idea of *having a lower bound* allows drawing a new link between grounding and fundamentality, in which objects and structures are on the same ontological floor and yet ultimately grounded in what I called higher structures.

The metaphysical benefits of WS will be discussed in detail in chapters 4 and 5, where this position will be applied to scientific and mathematical structuralism respectively.

[78] Bliss and Priest (2018) deem these objections not well-motivated and concerning *causal*, rather than *metaphysical* explanation. However, I am here presupposing that these worries should be considered, also because they are connected with some typical criticisms to (not foundationalist) structuralist ontologies, i.e. those raised by MacBride (2006) who argues that relations presuppose the individuality of objects, rather than grounding it, and Lowe (2012) according to whom a coherent structuralist ontology should include at least some self-individuating entities.

4. Weak Scientific Structuralism (WSS) and Quasi-Thin Physical Objects

In chapter 3, I outlined Weak Structuralism (WS) as a promising conceptual framework which combines the explanatory advantages of non-foundationalism with some intuitions of the received view. In this chapter, I am going to apply this position to scientific structuralism, defining it Weak Scientific Structuralism (WSS). Before doing so, I will investigate the metaphysical commitments of Ontic Structural Realism (OSR) and its main varieties. In the scientific structuralist literature, the relationship between objects and structures has been accounted for in terms of supervenience and ontological dependence. Ontological dependence, in particular, turns out to be a suitable notion to make sense of the metaphysical claims at hand in OSR. French (2010) has submitted a taxonomy of the main OSR views in terms of dependence, showing that *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR* (illustrated in chapter 1) can be articulated by referring to different forms of dependence (see chapter 3).

Significantly, ontological dependence has some features which suggest a non-eliminative approach to objects, thus ruling out *Eliminative OSR* – which is subject to the serious 'relation without *relata*' objection (chap. 1, sec. 1.2.1) – as a meaningful interpretation of Quantum Mechanics (QM). However, in chapter 1 (sec. 1.2.2) I argued that *Priority-based* and *Moderate OSR* presuppose a very *thin* conception of object, which is not immune from variations of the 'relation without *relata*' objection. In this respect, I will develop *Weak Scientific Structuralism* (WSS) as a further variety of OSR. WSS, consistently with the broader metaphysical picture outlined in chapter 3 (sec. 3.2.2), is expressed in terms of grounding, and in particular in terms of *Mutual Grounding* – which in this context will be applied to the physical objects and physical structures OSR is concerned with. Some general advantages of adopting grounding – rather than dependence – in order to investigate the structure of reality have already been presented in chapter 3. More specific benefits of a grounding-based interpretation of OSR will be advanced in this chapter, showing that grounding fits better with the prospects of securing the explanatory role that structures play in OSR. *Mutual Grounding* allows re-considering the relationship between objects and structures and to introduce quantum particles as *quasi-thin physical objects*. Quasi-thin physical objects – to be distinguished from *thin objects* as understood in both *Priority-based* and *Moderate-OSR* – seem to provide a strategy to avoid the 'relation without *relata*' objection. At the same time, *Mutual Grounding* – despite being a *reciprocal* relation – is consistent with the idea that structures are ultimately prior to objects.

4.1. The metaphysical commitments of OSR

OSR, as opposed to Epistemic Structural Realism (ESR), is a *metaphysical* thesis entailing the interpretation of structures as ontologically self-subsistent entities and the metaphysical decomposition of objects in purely structural terms. As I showed in chapter 1, both the relationship between structures and objects and the nature of objects have been differently characterized in the main varieties of OSR, which I defined *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR*.
Accordingly, these views are associated with different metaphysical theses, which it is worth taking into account.

Let us consider *Priority-based OSR*, which offers a useful starting point to evaluate the metaphysical import of OSR. *Priority-based OSR* admits both structures and objects in the inventory of what there is. Still structures bear all the ontological weight. This results in the idea that structures are *fundamental* and *prior* to objects. Concerning the *fundamentality* of structures, *Priority-based OSR* is committed to the claim that that structures – not objects – belong to the ontological fundamental level. As nicely put by McKenzie (2014, p. 353):

> Even if there should exist a set of 'fundamental objects' […] it is not the case that such objects would qualify as truly fundamental. Rather, in their view, the most basic metaphysical level of the world is constituted solely by the *structures* that our best physical theories describe.

The *priority* of structure is typically expressed by saying that putative physical objects are dependent, in a fashion yet to be clarified, upon relevant structures. For examples, French (2010, p. 104) takes OSR as committed to the view that «each fundamental physical object depends on the structure to which it belongs». In similar vein, Ladyman and Ross (2007, p. 130) say that «according to OSR (ontic structural realism), even the identity or the individuality of objects depends on the relational structure of the world». On this interpretation, *Priority-based OSR* appears to be committed to two theses, which I shall call the *Fundamentality Thesis* and the *Priority Thesis* respectively.

Fundamentality Thesis. All fundamental physical entities are structures.

Priority Thesis. Fundamental structures are prior to putative physical objects if these exist.[79]

When it comes to the articulation of *Eliminative OSR*, it is important to observe that this view entails a reductive reading of the *Priority Thesis*. In fact, recall that *Eliminative OSR* has no room for objects even at a non-fundamental level. Hence, while the *Fundamentality Thesis* is maintained, the *Priority Thesis* amounts to some form of ontological reduction. On this interpretation, the *Priority Thesis* can be rephrased as a *Reduction Thesis*: putative fundamental physical objects ontologically reduce to structures. On this view, objects can be regarded as empty places of the priority relation in question.

The *Fundamentality Thesis* rules out that objects are fundamental entities. However, some ontic structuralists favour a more moderate interpretation of the *Fundamentality Thesis*, one according to which only *some* fundamental physical entities are structures and others are objects. This is the case for *Moderate-OSR,* which replaces the *Fundamentality Thesis* and the *Priority Thesis* with a *Parity Thesis*, according to which structures and objects are ontologically on a par, or equi-fundamental. Thus, this view is not an eliminative one. Yet it is not a priority-based either, for objects and structures *symmetrically* depend on each other.

Any form of OSR should find a clear articulation of the *Fundamentality Thesis***,** the *Priority Thesis* and their variations, i.e the *Reduction Thesis* (rephrasing the *Priority Thesis* in *Eliminative OSR*) and the *Parity Thesis* (rephrasing both the *Priority Thesis* and the *Fundamentality Thesis* in *Moderate OSR*). In what follows, I will consider an interpretation of them in terms of dependence, which appears more suitable than supervenience. In sec. 4.2. I will refer to grounding and its advantages to examine the metaphysical commitments of Weak Scientific Structuralism in light of the considerations put forward in chapter 3.

---

[79] The *Fundamentality Thesis* and the *Priority Thesis* rephrase in a structuralist sense the fundamentality assumption (there is something fundamental) and the hierarchy assumption (reality is hierarchically structured by ontological dependence and grounding relations which are anti-symmetric, anti-reflexive, and transitive), which Bliss and Priest (2018) attribute to Metaphysical Foundationalism (MF). In *Priority-based OSR*, the *Fundamentality Thesis* takes structures to be the fundamental entities of reality, while the *Priority Thesis* states that structures are prior to objects in the sense that only structures belong to the truly fundamental level, whereas objects belong to a less fundamental one.

4.1.1. OSR and dependence

According to French (2010, p. 98) «there is a lack of clarity regarding the relationship between objects and structures, and it is also one that effects a separation between the eliminativist and non-eliminativist forms of ontic structural realism». To the aim of filling this gap, some ontic structuralists have invoked the notion of *supervenience*. For example, recall that Ladyman and Ross interpret OSR as the view that the world has «an objective modal structure that is ontologically fundamental, in the sense of not supervening on the intrinsic properties of a set of individuals» (2007, p. 130). The problem with supervenience is that that this notion is too coarse-grained for capturing the priority of structures over objects, and it does not guarantee an explanatory connection between them. Supervenience tracks only a modal covariation between certain things, thereby leaving us in the dark about why it holds. For example, McKenzie (2014, p. 357) argues that:

> supervenience is not at all explanatory of any relationship between the sub- and supervenient relata; it is often regarded as at best an indication that it is worth looking for an explanation of the evident connection between them, while not explanatory of it.[80]

By contrast, ontological dependence captures more precisely the relationship between objects and structures. In fact, as suggested in chapter 3 (sec. 3.1.1), ontological dependence is a metaphysical and *explicative* notion which expresses a non-causal priority relation between things.

Significantly, different forms of dependence elucidate different forms of OSR, providing a taxonomy in which *Priority-based OSR*, *Eliminative OSR* and *Moderate OSR* are distinguished.

French (2010, p. 105) describes *Priority-based OSR* as follows:[81]

1) The identity of the putative objects/nodes is (asymmetrically) dependent on that of the relations of the structure.

[80] Wolff (2011) raises a different objection to supervenience, focused on the link between reduction and supervenience: «for A to reduce to B, A has to supervene on B. For A to supervene on B, there *cannot* be a change in A without a change in B» (ibid. p. 611). However, the modal force carried by supervenience is too weak to establish such a reduction claim; moreover, counterexamples from physics shows that supervenience does not hold in the case of representations and symmetry groups. Consequently, representations do not reduce to symmetry groups.

[81] French (2010) defines this view Weak Structural Realism (WSR).

Priority-based OSR understands the relation between objects and structures *asymmetrically* (objects are admitted in the ontology, though as less fundamental than the structures they belong to) and focuses on the notion of identity, which on this view is understood *contextually* and defined with respect to weak PII (see chapter 1, sec. 1.1.3). According to French, the asymmetrical notion of dependence at play is adequately captured by Lowe's (2005) *identity dependence* (ID) which, as previously mentioned, is asymmetrical and, as such, vindicates the *Fundamentality Thesis* and the *Priority Thesis* which are crucial in *Priority-based OSR*.

ID: *x* depends for its identity upon *y* $=_{df}$ there is a two-place predicate "*F*" such that it is part of the essence of *x* that *x* is related by *F* to *y*.

Let us turn to *Eliminative OSR*. On this view, there is no question concerning the identity of objects as there is no room for objects in the ontology: «what exists then are not objects in any ontological sense» (French 2010, p. 106). Fairly obviously, there is no question concerning the existence of objects either – they are not genuine structure-occupants or *relata* of relations. Since *Eliminative OSR* denies the ontological robustness of objects, it is questionable whether a dependence relation applies at all. However, one could follow French (2010) and grant that objects and structures stand in a merely *conceptual* relation, according to which objects are interpreted as useful theoretical constructs which are entirely reduced to their structural features (in accordance with the interpretation of the *Priority Thesis* as a *Reduction Thesis*):

> Our putative objects only exist, in a sense, if the relevant structure exists and the dependence is such that there is nothing to them—intrinsic properties, identity, constitution, whatever—that is not cashed out, metaphysically speaking, in terms of this structure. […] this account amounts to claiming a form of mere conceptual dependence between objects and structure. (French 2010, p. 106).

If this is the case, then the following definition of *Eliminative OSR* can be provided:

2) the very constitution (or essence) of the putative objects is dependent on the relations of the structures.

This intuition is clarified by Fine's (1995) rigid essential (existential) account of dependence (EDER), which captures adequately the reformulation of the *Priority Thesis* as a *Reduction Thesis.* Objects solely exist (i.e. as *theoretical constructs*) if the relevant structure exists and there is nothing to them (identity, constitution, etc.) which can be defined independently of the structure:

EDER: *x* depends essentially existentially upon *y* $=_{df}$ it is part of the essence of *x* that *x* exists only if *y* exists.

Lastly, *Moderate OSR* introduces a symmetrical relation of dependence between objects and structures, that are ontologically on a par. As explained by French (2010, p. 104):

3): the identity of the objects/nodes is (symmetrically) dependent on that of the relations of the structure and *viceversa*.

French expresses this conception by the (rigid) modal-existential account of dependence (EDR), which allows for symmetrical relations [82] and is laid out as follows:

EDR: *x* depends for its existence upon *y* $=_{df}$ necessarily, *x* exists only if *y* exists.

However, the modal-existential account of dependence is subject to several counter-examples (chap. 3, sec. 3.1.1). Moreover, this account does not capture the essential/identity dimension that is clearly involved in OSR. For this reason, Wolff (2011) suggests to interpret *Moderate OSR* in terms of essence, and specifically in terms of Fine's (1995) notion of *constitutive essential dependence*. In fact, costitutive essential dependence admits symmetric dependence in the case in which objects are simultaneously defined. On this basis, Fine distinguishes this form of dependence from *priority* – which is clearly asymmetrical:[83] «this will lead to a mutual dependence, without priority, which might well be what moderate structural realists intend» (Wolff, 2011, p. 15). Differently put, consti-

[82] Tahko and Lowe (2020, sec. 2.1.) consider as an example the relation between Socrates and its life, which are said to be dependent on each other.

[83] In introducing non-circular simultaneous definitions, Fine claims that «given mis cycle-tolerant notion of dependence, priority can then be defined as the non-reciprocal case» (Fine, 1995, p. 284).

tutive essential dependence fits well with the *Parity Thesis* which is implicitly endorsed by *Moderate OSR*.

4.1.2. Supporting a non-eliminative approach towards objects

A reflection on ontological dependence itself allows us evaluate the tenability of each form of OSR. Wolff (2011, p. 2) discusses the role of ontological dependence in scientific structuralism and argues that «only certain forms of structural realism can be articulated using ontological dependence». Significantly, ontological dependence cannot serve the purpose of accounting for an eliminative interpretation of the relation between objects and structures. Ontological dependence is not an eliminative relation and requires that both the *relata* of the relation (objects and structures) should exist. To say that *x* ontologically depends on *y* means that *y* is *prior* to *x*, which is less fundamental than *y*; but this does not mean that *y* is to be eliminated: «unlike supervenience, ontological dependence is an explanatory relation and unlike a reduction claim, a claim of ontological dependence does not eliminate one of the *relata.* (Wolff, 2011, p. 12)».

By these means, the appeal to ontological dependence – which appears to be an adequate and fine-grained tool to account for the relationship between objects and structures – favours non-eliminative views, thus leaving us with *Priority-based OSR* and *Moderate OSR*. According to Wolff (2011), *Priority-based OSR* – as captured by Lowe's (asymmetrical) identity dependence (ID) – appears to be the most compelling alternative to eliminative OSR:

> Of the three versions of ontic structural realism discussed at the beginning [the three forms recognized by French, 2010, *Ed*.] only thin-object OSR comes close to being articulated using essential dependence as the relation between objects and structure» (Wolff, 2011, p. 16).

In fact, *Moderate OSR* meets with some difficulties even if it is interpreted according to the Finean notion of constitutive essential dependence: while in a structuralist framework it is plausible to say that an object is defined by the *role* it plays in a structure, and then by the relations of the structure, the other way round seems quite odd, for standardly roles in a structure are determined by the structural relations, and not by the objects occupying them (Wolff, 2011, p. 15).[84] Even more seriously, a symmetric notion of dependence clashes with the standard idea that explanations are asymmetric (see Lowe, 2003). Of course, an alternative is to understand the relationship between objects and structures as a mere *conceptual* relation (Esfeld and Lam, 2011) where relations are the *modes* in which objects exist (see chapter, 1, sec. 2.1.2). Still, on this re-elaboration of *Moderate OSR*, objects are actually identified with the relations in which they stand and it is unclear how Moderate *OSR* differs from *Eliminative OSR*.

Similarly, as mentioned in chapter 1, *Priority-based OSR* is questionable for the conception of *thin objects* it is committed to, which turn out to be *too weak* and *too structurally defined* to be admitted in the structural ontology and to be properly distinguished from the structure they belong to – thus being subject to variations of the 'relation without *relata*' objection affecting *Eliminative OSR*. Moreover, on this view, other worries concern the individuation of the relevant physical structures; as observed by O' Conaill (2014, p. 288) «French does not clarify whether the identities of the relations are to be understood as individual essences or as the relation of identity applied to each of the (structural) relations». In other words, the very identity of physical structures, on which they identity of objects depends, remains to be spelled out.

This is related to the problem of establishing which structures (concrete or abstract) determine objects for their identity:[85] if we assume that the relevant structures are *concrete*, as OSRists tend to suggest, then the concreteness of structures depends on the spatio-temporality of the objects constituting them. However, each physical structure can be abstracted in a corresponding mathematical structure, with objects being mere nodes or positions within them. So, the problem with *Priority-based OSR* (and also with *Moderate OSR*) is that if physical objects are entirely reduced to their structural features, then they are indistinguishable from their mathematical counter-parts. Therefore, «the difference between them must be a matter of one kind of object having either a structural feature or a non-structural feature which the other lacks [...]. (O'Conaill, 2014, p. 293)». Arguably, physical objects and their corresponding mathematical objects share the totality of their structural relations, so we need to ground their distinction in a non-structural property. Doing so, though, seems to undermine the *Priority-based OSR*'s claim that physical objects entirely depend for their identity on the structural features of the concrete structures they belong to. On this basis, O'Conaill presents

---

[84] This formulation is due to the fact that Wolff (2011) characterizes the relationship between objects and structures by taking into account structures, the positions/roles within them and the objects which may occupy these positions.

[85] Recall that this is one of the standard objection to OSR, according to which it is committed to the unpleasant result that physical and mathematical structures should be identified.

three possible strategies to understand the relationship between concrete and abstract structures: i) taking spazio-temporality to be a structural property (French, 2006); ii) claiming that concrete structures, but not mathematical ones, have an inherently modal-causal character (French, 2010; Esfeld, 2009); iii) denying that any distinction between concrete and abstract structures obtains (Ladyman and Ross, 2007). Each option raises further problems for *Priority-based OSR*: first, it is not obvious how to characterize spazio-temporality structurally – against (i).[86] Second, similar difficulties arise when it comes to define the causality of structures, for causal efficacy is intuitively a non-structural property of structures – against (ii). Third, identifying concrete and abstract structures raises further technical difficulties and seems to run counter OSR, which requires concrete structures to be an interpretation of the physical world – against (iii).

In the remainder of the chapter, I will develop an alternative conception of objects, by recalling WS as outlined in chapter 3. As I am going to show, the application of WS to the more specific context of OSR – introducing what I shall define Weak Scientific Structuralism (WSS) – has several advantages: first, the appeal to the notion of grounding, rather than dependence, captures more precisely the metaphysical commitments of OSR. Second, the interpretation of objects as *quasi-thin objects* – endowed with both structural and non-structural properties – provides an alternative strategy to distinguish physical structures from the corresponding mathematical structures (i.e. arguably, symmetry groups of group-theory) without renouncing the claim that the identity of objects is determined by the structures. Third, the notion of *shared structure* (Landry, 2007) suggests a possible way to individuate physical structures by means of symmetry groups of group theory. This notion, which has been introduced in the context of a semantic view of scientific theories, has interesting applications in WSS, where it provides a compelling strategy to conceptualize WSS's interpretation of fundamentality. Fourth, quasi-thin physical objects seem to escape the 'relation without *relata*' objection.

[86] For a more extensive understanding of this problem, related to the substantivalist-relationist interpretations of space-time, see O' Conaill (2014, sec. IV).

4.2. WSS and Mutual Grounding

As anticipated, one of the main problems of OSR is the elimination of objects from the fundamental ontology of the world – in *Eliminative OSR* – or, alternatively, the reduction of objects to their structural features – in *Priority-based OSR* and *Moderate OSR,* where objects are either mere nodes somehow emerging from structures or bearers/bundles of relations. Weak Structuralism (WS) can be adapted to the scientific structuralist framework as a response to these difficulties. In fact, as shown in chapter 3, WS relies on a relation of *Mutual Grounding* which naturally supports the introduction of more substantial objects in the ontology. Before examining this relation, and the different claims it is articulated in, it is worth to dwell on the merits of grounding in the context of OSR. The main advantage of invoking grounding to elucidate OSR concerns the link between grounding and explanation. This connection permits to safeguard the explanatory priority of structures in making sense of derivative physical objects – objects are explanatorily dependent on structures. If structures ground physical objects, and if *groundees* (what does the grounding work) explain or help explaining what is grounded, then structures explain or help explaining physical objects. As shown in chapter 3, the complete story of how to relate grounding and explanation is more complicated than this. Still, the principles of 'inheritance' and 'involvement' (see sec. 3.1.3) suffice in this framework to account for the link between grounding and explanation, which is plausibly stronger than that presupposed by dependence (cf. Tahko, 2015) and supervenience (which lacks a clear explanatory import). In fact, the very idea of grounding is motivated, at least partially, by the observation that this notion underlies a variety of claims where something holds *because* or *in virtue of* another. A second merit of grounding concerns its connection with the notion of *metaphysical determination*. McKenzie (forthcoming) argues that ontological dependence does not ensure determination. What grounds this worry is the possibility of dependence without determination (McKenzie forthcoming, pp. 10-11). As an example, McKenzie considers the possible entanglement relation between two electrons. If we assume, taking into account the 'relation without *relata*' objection, that relations cannot exist without *relata*, then this seems to be a case where the entanglement relation would be dependent on the two electrons, but it would be incorrect to argue that the electrons (and their properties) determine the entanglement relation. To give another example, one might think that particles are dependent on spacetime symmetries without the latter determining the former.[87] By contrast, grounding *is* a constitutive form of determination or, as Fine puts it, a «determinative connection» (2012, p. 37). Therefore, grounding is preferable to dependence in OSR because it blocks the problematic possibility of dependence without determination.

As argued by Wolff (2011), one of the key features of ontological dependence is to be a non-reductive notion that, as such, is suitable to describe non-eliminative OSR-views. Is grounding non-reductive as well? As suggested in chapter 3, sec. 3.1.2. the relationship between grounding and reduction is controversial. However, some authors (Audi, 2012; Tahko, 2015) – contradicting the idea of grounding as delivering an *ontological free lunch* – have suggested that grounding is non-eliminative, and that the grounded entities/facts are an addition of being over and above the facts/entities they are grounded in. Therefore, it seems plausible to take grounding, like dependence, as a non-eliminative notion which fits well with non-eliminative interpretations of OSR.

Given the advantages that grounding brings to OSR, we are in the position to recall its non-foundationalist reformulation – *Mutual Grounding* – advanced by Weak Structuralism. In fact, *Mutual Grounding* as defined above (i.e. if x grounds$_{(I)}$ y, then y grounds$_{(E)}$ x) re-conceptualizes the relationship between quantum particles and physical structures in a way that allows us to individuate a new category of objects – *quasi-thin physical objects* – distinguished from the conception of objects presupposed by both *Priority-based OSR* and *Moderate OSR.* Let us define this view *Weak Scientific Structuralism* (WSS). The main features of quasi-thin objects will be illustrated by investigating their structural and non-structural properties and by providing two more specific definitions of quantum particles as *quasi-thin physical objects*. In particular, along the lines of Wolff (2011), non-structural properties of quantum particles will be identified with their *kind properties* – distinguishing electrons, muons, etc. On this basis, the two main claims of *Mutual Grounding* (*Object Identity* and *Structure Existence*) will be cashed out with respect to quantum particles and physical structures, claiming that quantum particles – when entangled – are grounded in quantum entanglement structures for their identity, while quantum entanglement structures are grounded in quantum particles for their existence. Assuming the difference between *Mutual Grounding* and symmetric grounding (chap. 3. sec. 3.2.2), objects and structures in WSS turn out to be mutually,

[87] However, it is not clear if this objection to ontological dependence concerns *all* forms of dependence illustrated in chapter 3 (sec. 3.1.1); for example, Tahko and Lowe take *identity dependence* (ID) – which is more fine-grained than modal-existential dependence – to express «the *determination* of the individuality of objects in terms of the individuality of other objects». (Tahko and Lowe, 2020, sec. 4.2, my emphasis*)*. In a similar vein, Tahko (2015, p. 100) claims: «to say that the identity of *x* depends on the identity of *y* is to say that *which* thing of its kind *y* is metaphysically determines *which* thing of its kind *x* is».

but not symmetrically grounded in each other. Significantly, in WSS, objects appear to be *substantial enough* to respond to the 'relation without *relata*' objection and yet *thin enough* to preserve the priority of structures, which will be spelled out in accordance with *Structure Identity* – the third grounding claim WSS is based on.

4.2.1. OSR's structural objects and quasi-thin physical objects

Let us now introduce quasi-thin objects by comparing them with objects as understood in *Priority-based* and *Moderate OSR*. The focus on these two views is motivated by the fact that *Eliminative OSR* does not admit objects at all in the ontology, which consists of structures only. Both *Priority-based* and *Moderate OSR* seem to suggest an entirely structural characterization of physical objects, one in which quantum particles are entirely defined by their structural properties.

This point is made clear by Busch (2003, p. 214):

> In insisting on the importance of structural properties over non-structural properties, it seems at first sight that the ontic structuralist has adopted a primacy view of second-order properties over first-order properties.

In other words, in OSR structural properties are primary and non-structural properties are secondary, thus contrasting the standard object-oriented metaphysics which takes non-structural properties and structural properties to be first-order properties and second-order properties respectively. A first issue concerns how to characterize structural and non-structural properties in OSR more precisely. A definition of structural properties has been already suggested in the context of mathematical structuralism. In fact, in chapter 2 (sec. 2.2.3), I showed that one way to specify structural properties of objects (i.e. the *invariance account)* is to refer to the properties which remain invariant under structure-transformations. Significantly, a similar definition can be applied in OSR, where structural properties of quantum particles (i.e. state-dependent properties such as *position* and *momentum*) are generally described as those properties which remain *invariant* under symmetry-groups transformations in group theory.

As specified by Ladyman (2020, sec. 4.1.):

> We have various representations of some physical structure which may be transformed or translated into one another, and then we have an invariant state under such transformations which represents the objective state of affairs.

However, this definition admits exceptions, consisting of the state-independent properties of quantum particles. McKenzie (forthcoming) takes *kind properties* as a prominent example:

> Let us first take it as a datum that the objects of physics come sorted into kinds, as characterized by a set of determinate values of determinable physical properties, such as determinate mass, charge, and spin. For brevity, let us refer to these determinate properties collectively definitive of kinds as the determinate kind properties. (McKenzie, forthcoming, p. 5).

On McKenzie's view, kind properties are part of the concept of object as opposed to the concept of structure, i.e. they set out the criteria for defining objects in physics. In a similar vein, Wollf (2011) refers to *kind properties* as the properties which qualify particles as electrons, muons, etc. Wollf (2011, p.9) specifically refers to *mass* and *spin* properties as non-structural, kind properties of quantum particles:

> How do we make sense of the difference between muons and electrons if not through the difference in mass between muons and electrons? Mass is a state-independent property, not a state-dependent property[...]. Different kinds of systems can be in a singlet state, and while it is true that neither particle can be said to be the spin-up or the spin-down particle, it makes a difference for physics what kind of particles are in a particular singlet state, e.g. whether they are electrons or muons.

Some clarifications concerning kind properties are needed, since they play a crucial role in the present discussion. A first issue concerns whether kind properties are reducible to a structural, group-theoretic characterization. The problem surfaces in the following passage by (Wolff, 2011, p. 16).

> The starting point for a structural account of properties like mass and spin is the classification by Wigner ([1939]) of particles as irreducible representations of the Poincarè group. [...] Wigner's classification reveals how the different kinds of particles belong to a common structure, the structure of the Poincare´ group, by showing how different kinds of (elementary) particles can be thought of as different irreducible representations of that group.

More recent attempts to apply this reduction have been performed by Castellani (1998) and Muller, (2009; 2011). For example, Muller (2011, p. 232) claims that «properties of elementary particles like mass, charge and spin-magnitude [...] are determined by symmetry relations, which makes them acceptable for the structural realist».

However, both Wolff (2011) and McKenzie (forthcoming) consider this strategy unsuccessful. Wolff (p.16) argues that it is unclear how symmetry groups can be concrete structures, «unless we take the particles to instantiate this structure, in which case we might worry that the particles are far from being eliminable». More importantly, even if we accept such structural analysis of kind properties, state-independent properties are hardly *reducible* to structures: «the different kinds of particles have different values for mass, and accordingly they have to be thought of as *different* irreducible representations of *the same* group» (Wolff, 2011, p. 17). As a consequence, representations are not reducible to symmetry groups and a structural account of state-independent properties by no means supports a reductionist reading of structuralism. Indeed, this poses another serious threat to OSR – to be added to the objections illustrated in chapter 1, sec. 1.2.1, 1.2.2 – and specifically to *Eliminative OSR*.

Another serious counter-example comes from McKenzie (forthcoming) in the context of Quantum Field Theory (QFT). McKenzie takes the fundamental structures to be the symmetry groups of QFT, such as the Poincaré Group, and objects to be ‘fundamental’ kinds such as boson kinds and fermion kinds. While specific local gauge symmetries uniquely determine the bosons kinds that there will be, their specific integer spin and that they are massless (in the case of unbroken symmetries), the symmetry groups do not entail «which kinds of fermions we can expect to be instantiated» because, unlike boson kinds, their determinate kind properties are not «similarly uniquely determined» (McKenzie, forthcoming, p. 18). For example, the relevant symmetry groups do not entail the determinate masses of fermions that we can expect to find in nature. McKenzie (p.) concludes that «thus claims made by structuralists that the ‘properties of elementary particles like mass, charge and spin-magnitude [...] are determined by symmetry relations [...]’ (Muller [2011], p. 232) would in fact seem to be simply untrue».[88]

More broadly, McKenzie's argument aims at showing not only that a reductionist (and thus eliminativist) interpretation of OSR fails, but also that *Priority-based OSR* in the idiom of determination «must be regarded as unfounded» (p.23). Kind properties are neither *reducible* to structural properties, as already pointed out by Wolff (2011), nor *describable* in structuralist terms, for there are counter-examples from physics – the case of fermions – where state-independent properties are not determined by the relevant symmetry groups. In other words:

> should the kind properties essential to fundamental objects prove [...] amenable to structuralist analysis, then, priority-based structuralism will be home and dry. Unfortunately, however, things are not so easy in the case of kind properties. On the contrary, it seems that these pose an obstacle to priority-based structuralism that we currently have no idea how to circumvent. (McKenzie, forthcoming, p. 18).

Along the lines of Wolff (2011) and McKenzie (forthcoming), I support the idea that kind properties, as state-independent properties, should not be analyzed structurally. However, as I am going to show in the next sections, this does not commit one to renounce structuralism, but rather to rethink the relationship of fundamentality between objects and structures consistently with the metaphysical picture suggested by WSS.

Second, there is the question whether kind properties are intrinsic properties. Intuitively, non-structural kind properties do not correspond to intrinsic properties, defined as the «properties that are independent of whether the object is alone or accompanied by other objects» (Esfeld and Lam 2011, p. 144): in fact, kind properties do not fix the identity of objects as individuals, but as ‘packaged’ into kinds (given by determinate correlations of mass, spin, charge). However, one has to acknowledge that the question of the 'intrinsicality' of kind properties is controversial; for exam-

[88] The remarks contained in this paragraph are taken from Bianchi and Giannotti (2021).

ple, McKenzie (forthcoming, p. 8) describes objects as follows: «the category of objects consists of pluralities of entities defined in terms of a shared set of determinate kind properties, where at least some of these properties are intrinsic properties». However, it is noteworthy that, on McKenzie's view, *some* – not *all* – kind properties are intrinsic. For example, even if the relevant symmetries do not determine the kind properties of fermions, we know that the latter are constrained significantly by the symmetries. For example, we know that the determinate properties of fermion kinds should be consistent with the possibilities admitted by the representations of the symmetry group in question. Moreover, symmetry considerations reveal that the fundamentality of specific structures and their associated laws impose a limit on the number of fermion kinds that can co-exist consistently. To use McKenzie's example, if the SU(3) gauge symmetry is fundamental, then there cannot be more than sixteen kinds of fermions for the theory to be still valid up to arbitrarily high energy scales: «this suggests that at least some of the essential properties of the fundamental kinds are not intrinsic properties» and also that «all fundamental kind properties may eventually be established as extrinsic» (p. 25).

Third, it is worth noting that properties such as mass, spin and charge are *essential* properties of quantum particles. This constitutes another strong to objection to OSR, in which essential properties are generally understood as structural properties. However, there is a sense in which kind properties are still *secondary* to structural properties and then acceptable for structuralists (and also consistent with the metaphysical features of WSS as a weak and yet structuralist position). Kind properties distinguish electrons, muons, etc. but, assuming two electrons in a singlet-state, they leave underdetermined which one is which. It follows that quantum particles are indistinguishable in a much stronger sense, and that solely the structure fixes their very identity – what they are as opposed to all the other objects in the same structure. This entails distinguishing two interpretations of identity: primarily, the identity of objects as *individuals* and, secondarily, the identity of objects as *kinds*. This point is implicitly suggested by Wolff (2011, p. 26):

> Particles *qua* individuals are thin objects. To the extent that we understand their identity as individuals, we understand it in terms of the state they are in. This leaves unaffected their 'kind identity', that is, their identity as electrons rather than muons. Which kind of particles they are does not depend on any particular state the particles are in.

Therefore, while it is true that objects, considered individually, are grounded in structures for their identity (particles of the same kind can be switched without entailing a difference for the overall state), there is at least one sense in which objects are not exhausted by the role they play in a structure – when objects are considered as belonging to kinds, corresponding to determinate correlations of mass, spin and charge: this is seems to be particularly emphasized by the following passage:

> Different kinds of systems can be in a singlet state, and while it is true that neither particle can be said to be the spin-up or the spin-down particle, it makes a difference for physics what kind of particles are in a particular singlet state, e.g. whether they are electrons or muons. (Wollf, 2011, p. 9).

In conclusion, despite the intrinsicality of kind properties is controversial, I think that there is fair amount of reason for adopting the view that kind properties are i) non-intrinsic, ii) essential (but still secondary to structural, state-dependent properties).

Kind properties so defined allow introducing quantum particles as *quasi-thin physical objects*, defined by the conjunction of structural, primary properties and non-structural, secondary kind properties: «we know that, for something to be mass, it will be mass of *something»*. (Busch, 2003, p. 214). On these grounds, quasi-thin physical objects can be distinguished from the weaker *thin objects* endorsed by *Priority-based OSR* and *Moderate OSR* – which, despite admitted in the ontology, are actually reduced to their structural properties, leaving the very distinction between objects and structures obscure: «if, as it turns out, structures are in fact the most basic constituents of the world, we need to know where structures cease to exist and objects start». (Busch, 2003, pp. 214-215). Two more specific definitions of quasi-thin physical objects will be suggested in sec. 4.2.3. For now, let us just observe that *quasi-thin physical objects* provide us with some tools to understand what objects could be in OSR and where to set the cut-off between objects and structures.

Still, quasi-thin objects confront us with two main issues, which will be addressed in due course:

i) are quasi-thin physical objects substantial enough to avoid resulting in a 'no-objects-at-all' position?
ii) Are quasi-thin physical objects weak enough to preserve a structuralist framework?

4.2.2. Objects and structures: a new relationship of fundamentality

Let us now develop Weak Scientific Structuralism (WSS) and reformulate the relationship between objects and structures in terms of *Mutual Grounding.* As I argued in chapter 3, *Mutual Grounding* relies on three main claims, which in this context will be applied to objects and structures in scientific structuralism.

1) *Objects Identity*: quantum particles are fully grounded for their *identity* (not for their existence) in the identity of the quantum entanglement structure they belong to.

2) *Structure Existence*: quantum entanglement structures are fully grounded for their *existence* (not for their identity) in the existence of the quantum particles constituting them.

3) *Structure Identity*: the identity of physical structures is fully grounded in the identity of the corresponding mathematical symmetry-groups.

Let us present these claims in more detail and start with *Object Identity*, which includes two different theses:

Object Identity (a): quantum particles are fully grounded in the relations of the structure for their *identity* .
Object Identity (b): quantum particles are *not* fully grounded in the relations of the structure for their *existence*.

Object Identity (a) is quite straightforward in the context of OSR; recall that, as entailed by the Indistinguishability Postulate (IP) in QM (chap.1. sec. 1.1.3.), quantum particles in a singlet-state are indistinguishable in isolation: they can be permuted while leaving the relevant state unchanged. Therefore, solely the whole quantum entanglement structure they are in grounds their identity.[89]

[89] Such view is generally supported by by a discussion of quantum entanglement in terms of non-separability (see Esfeld, 2004).

Nevertheless, as the 'relation without *relata*' objection shows, relations of quantum entanglement require *things* to stand in the relations: Esfeld (2004, p. 613) argues for a non-eliminative metaphysics of relations for quantum particles as follows:

> «relations require things that stand in the relations (although these things do not have to be individuals and they need not have intrinsic properties)».

This idea is nicely captured by *Object Identity (b),* positing *entities* which may exist independently of the structure they belong to in virtue of their non-structural, kind properties. One could wonder how to characterize the existence of objects whose identity is given by the structure: if the identity of quantum particles is structurally defined, what does their existence – conceivable independently of the structure – exactly amount to? I submit the existence of objects is not reduced to their essential structural properties, since it also results in their *non-structural properties* which – as suggested by Wolff (2011) and McKenzie (forthcoming) – can be identified with *kind properties*. The understanding of quasi-thin objects requires the distinction between *objects* on the one hand and *entities* on the other hand: to be an *entity* is to be the subject of a predication of properties. This is not equivalent to being an *object* or an *individual*, for which further requirements need to be fulfilled (having an intrinsic identity or a 'primitive thisness').[90] On this view, it is not implausible to consider entities which are not objects, i.e. *properly individuated entities.* Recall that kind properties are able to distinguish electrons, muons, etc., thus fixing the identity of quantum particles as *kinds* – on these grounds, quantum particles appear to be *entities*. However, as above, kind properties cannot determine the *identity* of quantum particles as *individuals*, telling us which one is which – quantum particles are not proper *objects*, endowed with a specific *identity* distinguishing them from other objects of the same kind. Still, kind properties provide us with some tools to account for the *numerical diversity* of, say, two electrons in a singlet-state, showing that – though indistinguishable – they cannot collapse in a single entity. First, some may claim that numerically distinct objects fall under the same kind as a fact about kind-membership. This idea is suggested by McKenzie (forthcoming) as follows:

> [...] Since it is presumably part of the concept of an object that it is a particular – that is, something essentially distinct from a universal – then it should, at least in principle, be possible for there to

[90] Cf. Keränen (2001, p. 313)

> co-exist numerically distinct tokens of the same kind. [...] That is, we will take it as a given that there are many electrons, many photons, many positrons, and so on (assuming for argument's sake that these are fundamental kinds): to do otherwise seems wholly too revisionary for a naturalistic thesis. (McKenzie, forthcoming, pp. 5-6).

Second, and more importantly, kind properties are related to the roles that particles play in physics. A deeper investigation into the roles of quantum particles suggests that two electrons in a singlet-state – despite sharing their mass, spin and charge values – are two and not just one; in fact, as it is well-known, the two electrons have equal but opposite spins. As introduced in chapter 1, this can be interpreted as a case of weak-discernibility (cf. Saunders, 2003; 2006), whereby the numerical diversity of objects is grounded in the symmetric and irreflexive relation "having opposite direction of each component of spin to". As claimed in chapter 1 (sec. 1.2.2.) and 2 (sec. 2.4.2) this interpretation is not without problems, for the irreflexive relation in question seems to already presuppose – rather than grounding – the numerical diversity of objects. However, the same situation can be read slightly differently, by shifting the focus on spin as a kind property of electrons. On this view, if we take kind properties to be connected to the roles that particles of a specific kind play, it is plausible to assume that two electrons in a singlet-state numerically differ at least for one kind property – their spin value, that is spin-up and spin-down for each particle respectively. The difference in the electrons' spin directions shows that they must be *two* in order to play their proper role in the singlet-state, even though we still do not know which particle occupies which position in the state.

Note that this strategy, on which much more should be said in an experimental framework, mirrors the solution to the identity problem affecting mathematical structuralism proposed in chapter 5, sec. 5.3.2. I acknowledge that this approach is far more controversial in physics; still, I am not here advancing a definitive solution, but simply gesturing at an open path that is worth exploring in order to ground the numerical diversity of objects independently of structures.

The argument here proposed is based on the distinction between *numerical diversity* and *identity*. In fact, one could wonder whether the numerical distinguishability of objects suffices for establishing their very identity. If so, this would commit one to the idea that the identity of quantum particles is non-structural, after all, with obvious problems for structuralism as broadly understood. However, as introduced in chapter 3, identity is often taken to involve the *individuality* of an object, i.e. what it really is as opposed to the other objects in the same plurality. In other words, identity deals with «*ontological* issues concerning the metaphysical basis of individuality». On the contrary, distinguishability is related to «*epistemological* issues concerning how we distinguish objects» (French, 2019, sec. 1.). On these grounds, French (ibid.) claims that «distinguishability and individuality should be kept *conceptually* distinct».[91] A similar point is stressed by Lowe (2009), who distinguishes two senses of individuation (which seem to correspond to identity and numerical diversity respectively): a metaphysical one, where «what 'individuates' an object [...] is whatever it is that makes it the single object that it is» and an epistemic one, that is for one «[...]to 'single out' that object as a distinct object of perception, thought, or linguistic reference» (Lowe, 2003, p. 75). In other words, while metaphysical individuation is an *ontological* relationship between entities, epistemic individuation is an epistemic or cognitive activity. The idea here suggested is that the identity of objects, i.e. their metaphysical individuation, is defined by the structure, while the numerical diversity of objects, i.e. their epistemic individuation, is determined by their non-structural kind properties, which are related to the roles of particles in QM.

Consider now *Structure Existence* which, as *Object Identity,* involves two distinct claims:

Structure Existence (a): quantum entanglement structures are fully grounded in quantum particles for their *existence*.
Structure Existence (b): quantum entanglement structures are *not* fully grounded in quantum particles for their identity.

In the context of scientific structuralism, *Structure Existence* (a) is motivated by the fact that structures require spatio-temporal objects to *exist* in the *physical* world and then to be distinguished from abstract, mathematical structures. In fact, the identification of concrete and abstract structures raises serious problems for OSR as a position which aims at describing our contemporary physics. In this picture, the distinction between concrete and abstract structures is based on non-structural kind properties of quantum particles, which define their *existence* independently of structures. Still, the appeal to non-structural properties does not commit one to renounce the claim that physical objects are grounded for their identity in structures – as claimed by O'Conaill (2014) – because in WSS this is stated by *Object Identity* (a); given the distinction between the identity and the existence dimensions, *Object Identity* (a) is consistent with *Structure Existence* (a).

At the same time, the identity of structures can be determined independently of objects themselves, in accordance with *Structure Existence (b).* In fact, some problems of individuating objects and structures symmetrically, as suggested by *Moderate OSR*, are outlined by Wolff (2011). The individuation of physical structures is troublesome in OSR. Here I take physical structures to be grounded for their identity in symmetry-groups which are relevant for the statistics of QM, thus introducing the third *Structure Identity* grounding claim. It is worth noting that this interpretation does not entail that physical and mathematical structures are identical. Recall that the *concreteness* of structures is established by *Structure Existence* (a). By contrast, what I mean here is that physical structures are *represented by* and *embedded in* their corresponding mathematical structures. This way of interpreting the relationship between physical and mathematical structures presupposes that there exists a *hierarchy* of structures, in which symmetry-groups occupy the higher, top-level position and in this sense constitute the structure (as relations between the physical relations that matter for QM) of a structure (as the relations between objects/nodes or positions): in other words, quantum entanglement structures are themselves describable in structural terms. As noticed by Roberts (2011), in principle there are even higher, more abstract structures to deal with: «there is an important sense in which symmetry groups are describable in terms of their own symmetry group structure»[92] (p. 57). However, an infinite regress is not desirable, so symmetry-groups can be understood as those structures which are the «least abstractly removed from the real world» (p. 63) or, alternatively, as the structures that do play an effective role for physics, whose identity can be taken as primitive.[93] The relationship between quantum entanglement structures and symmetry groups of group theory should be spell out. Landry (2007, pp. 5-6) refers to the concept of *shared structure*, which can be applied in this context in order to account for the relationship between quantum structures and higher symmetry groups: «it is enough to say that, in the context under consideration, there is a morphism between the two structured systems (mathematical or physical) that makes precise the claim that they share the appropriate kind of structure».

Differently put,

[92] As specified by Roberts, p. 57, «The 'symmetry group structure' describing a group G itself is called the automorphism group, Aut G».

[93] For example, the rotation group SO(3) plays a crucial role in the understanding of *momentum* in quantum systems.

> The relationship of isomorphism between the different levels of the hierarchy should be relaxed to one of "shared structure", of which isomorphism is a special case, but which includes weaker relationships. The inclusion of these weaker relationships means that the structural realist must demonstrate the relationship of shared structure on a case-by-case basis, showing that the relevant structure from the high level theoretical model (about which we are sup- posed to be realist) transfers down the hierarchy appropriately. (Branding, 2011, p. 56).

Of course, much more should be said on the relationship between QM and group theory. For reasons of space, I cannot here reconstruct the technical physical details. For a more extensive overview on the role of group-theory in QM, and some specific examples, see French (1999). What is important to observe in the framework of Weak Scientific Structuralism (WSS) is that, on this view, it makes sense to say that two physical structures are identical if they share the same mathematical structure or, equivalently, if they can be abstracted in the same mathematical structure, thus vindicating *Structure Identity* introduced above.[94]

4.2.3. Quantum particles as quasi-thin physical objects

One of the main outcomes of WSS and the relation of *Mutual Grounding* concerns the introduction of (more substantial) quasi-thin physical objects in the ontology, to be distinguished from the entirely structural objects (also defined as *thin objects*) which are at play in other forms of non-eliminative OSR.

In particular, two more specific definitions of quantum particles as quasi-thin physical objects emerge from the combination of claims *Object Identity* (a; b) and *Structure Existence* (a; b):

1. A. *Quasi-Thin Objects [Existence]*: Quasi-thin physical objects are entities whose essential *identity* is grounded in the relevant structure, but whose existence is necessary to posit rela-

[94] For a different approach on the relationship between quantum entanglement structures and symmetry-groups, see French (1999) French and Ladyman (2003) and French (2014) who, by endorsing a semantic view of scientific theories, introduce the set-theoretic notions of *partial isomorphisms* and *partial structures* to formally express the idea of *shared structure:* «[t]hus the appropriate model-theoretic formulation would be one involving par- tial structures in general or partial function spaces in particular [19] and that the relations between the corresponding structures would consequently be those of partial isomorphism. Furthermore, each theory, group theory and quantum mechanics, is itself structured, in the [set-theoretical] manner indicated above» (French, 1999, p. 201). Still, this view has been criticized by Landry (2007) and Branding and Landry (2011) and for the present discussion the notion of *shared structure* suffices to understand the 'bridge' between physical and mathematical structures.

tions themselves.

2. A. *Quasi-Thin Objects [Kind]*: Quasi-thin physical objects are entities that (in addition to their structural properties) possess also secondary, non-structural kind properties (the properties that qualify quantum particles as electrons, muons, etc.) which suffice to fix their *numerical diversity*.

Definitions 1.A and 2.A allow introducing quantum particles as *entities* whose existence is conceivable independently of structures. As suggested, *entities* should be distinguished from *objects* as properly individuated entities. Quasi-thin physical objects so understood differ from objects in both *Priority-based OSR* and *Moderate OSR*. On the one hand, quasi-thin objects are defined for their very identity by the structure – exactly as thin objects in *Priority-based OSR* and *Moderate OSR*. On the other hand, quasi-thin objects are *countable* before entering in the relations, for their numerical diversity is non-structurally defined by their state-independent kind properties as a fact about the roles that particles play in physics. This avoids some difficulties of both *Priority-based OSR* and *Moderate OSR*. In *Priority-based OSR*, symmetric and irreflexive relations (i.e. having opposite direction of each component of spin to...) are supposed to account for the numerical diversity of objects in the context of a weak version of PII. However – as MacBride (2006), among others, observed – such relations actually presuppose numerically distinguished objects in order to obtain and then to confer identity on the *relata*. In *Moderate OSR*, by contrast, the numerical diversity of objects is assumed as primitive, with obvious difficulties because the idea of a primitive numerical diversity echoes the controversial concepts of primitive thisness and haecceity – which, in a naturalist stance, can be seen with suspicion as being obscure and objectionable. Before moving on, a clarification is needed, in order to specify why assuming the numerical diversity of objects as a fact about kind properties does not amount taking it as primitive. First, as suggested above, kind properties are intended to individuate objects just epistemically, in the sense specified by Lowe (2003). Put differently, it is not really the metaphysical identity of objects that is at stake here, bur rather the lighter epistemic notion of numerical diversity, or countability, that can be helpful when it comes to singling out indistinguishable objects as distinct objects of thought. So understood, numerical diversity clearly differs from a thicker, metaphysical notion of primitive identity – the notion that is most troublesome in scientific structuralism, as claimed by Morganti (2010, p. 232): «it appears useful to restrict the label 'primitive thisness' or 'haecceity' to primitive intrinsic identity when intended as a 'thick' metaphysical property, truly additional to other properties of things».

Second, kind properties help establishing the numerical diversity of objects for reasons related to the roles that particles play in quantum entanglement states, something that should be settled by experiment, rather than explained in the context of a primitive notion of identity.

Third, the plurality of entities of the same kind fits well with a naturalistic conception of physics, for, as pointed out by McKenzie (forthcoming), it would be simply too revisionary to deny that there are many electrons, many photons, etc. Moreover, «it is not the plurality requirement that causes the problems for structuralism, but rather the requirement of intrinsic kind properties» (McKenzie, forthcoming, p. 5).

In the next section, I will consider whether WSS and quasi-thin objects escape the 'relation without *relata*' objection without being at odds with a structuralist approach.

4.2.4. A possible response to the 'relation without *relata'* objection

Let us reconsider the two main question raised by quasi-thin objects:

i) are quasi-thin physical objects substantial enough to avoid resulting in a 'no-objects-at-all' position?

ii) are quasi-thin physical objects weak enough to preserve a structuralist framework?

First, as opposed to eliminative OSR, a (weak) notion of object is re-established: quasi-thin objects are not entirely reduced to their structural features, since they are endowed with both structural and non-structural *kind properties.* Such properties define them – if not as objects in a proper sense – as *entities* which are numerically distinguishable and conceivable independently of the structures. On these grounds, quasi-thin objects are *substantial enough* to be admitted as legitimate *relata* of the structural relations, in accordance with the idea that relations need some things to be related with each other (the first *i.* condition is satisfied). This suggests a plausible response to the 'relation without *relata*' objection affecting OSR, for quasi-thin objects as defined by definitions 1.A and 2.A are not entirely decomposed in structuralist terms: some of their state-independent properties – those classifying quantum particles in different kinds – are preserved in the picture, despite as *derivative* ones. As claimed by Wollf (2011, p. 16) «quantum particles depend on the structure for their identity and *nothing else*», meaning that quantum particles are something over and above their structural features. Therefore, unlike *Priority-based OSR* and *Moderate OSR*, WSS is an explicit non-eliminative structuralist position and, as such, does not end up being a form of *Eliminative OSR*.

Second, consistently with OSR, quantum particles do not possess intrinsic properties, which would commit one to the object-oriented metaphysics that scientific structuralism wants to contrast. In fact, it is widely held in QM formalism that quantum particles – when entangled – lack any quantum pure state, which is exhibited just by the whole entangled system. In such cases, quantum particles are devoid of any properties that may characterize them individually. However, this is not always the case; at least on some interpretations of QM (i.e Copenhagen-type interpretations) quantum particles are not necessarily in entanglement states, and then display a pure state in isolation. So, it is worth emphasizing that the 'quasi-thinnes' of quantum particles entails a specific two-place relation, involving a particle and a time. Taking into account these two possibilities (quantum particles as entangled/non-entangled), I focus on the more standard situation described by OSR, in which quantum particles are in fact entangled and then are grounded in the structure for their identity. In this respect, quantum particles are secondary to the structures (quasi-thin objects are *weak enough,* thus responding to *ii.*)

To sum up, such objects appear to be something more than the 'no-object at all' OSR is committed to (whether eliminated *tout court* or understood as thin, entirely structural objects), but something less than the 'thicker' objects in opposition to which OSR has been originally introduced.

4.2.5. Taking Stock: WSS as a middle-ground approach

Let us now evaluate whether *Weak Scientific Structuralism* (WSS) satisfies the main properties attributed to Weak Structuralism (WS) in chapter 3 and – if so – how they can be applied to quantum particles and physical structures relevant to scientific structuralism.

WSS ensures that objects and structures are on a par, for they are grounded in each other in the sense specified by *Mutual Grounding*. In other words, quantum particles and physical structures are equi-fundamental. On this basis, they cannot be well-founded in a standard foundationalist sense, i.e. being finitely grounded. However, the distinction between *Object Identity* and *Structure Existence,* involving the identity and the existence dimension respectively, makes objects and structures mutually (M), but not symmetrically (¬AS) grounded in each other. This motivates the reference to *Mutual Grounding* as a separate notion, ultimately different from symmetric grounding. *Mutual Grounding* – as opposed to symmetric grounding holding for example in Coherentism – is in principle compatible with anti-reflexivity (AR), for each grounding relation (*Object Identity*, *Structure Existence* and *Structure Identity*) is asymmetric on its own and then does not lead back to the starting point. Significantly, *Object Identity* and *Structure Existence* allow understanding WSS as a further interpretation of OSR, to be distinguished from the OSR-views I illustrated above.

Quite obviously, WSS differs from *Eliminative OSR*: by introducing quasi-thin physical objects, WSS rejects clearly an eliminativist approach, for not only objects are admitted in the ontology, but also they possess both structural and non-structural properties – they cannot be reduced to structures, thus contradicting the *Reduction Thesis* endorsed by *Eliminative OSR*.

Let us now consider *Moderate OSR*; apparently, WSS and *Moderate OSR* are comparable, since they both entail a form of the *Parity Thesis* (objects and structures are equi-fundamental). Still, the mutuality of the relation – along with the two directions, i.e. *identity* and *existence*, in which the grounding claims go – gives us compelling reasons to take the two views as distinct. In fact, *Moderate OSR* understands objects and structures as *symmetrically* dependent on each other, with no further senses or dimensions in which the dependence claim can be read.

At the same time, WSS does not correspond to *Priority-based OSR*, claiming that physical structures are prior to the quantum particles constituting them: in WSS objects and physical structures belong to the same fundamental level and then the *Priority Thesis* assumed by *Priority-based OSR* does not hold at this level. However, there is another level at which the *Priority Thesis* obtains. As shown by the *Structure Identity* claim, the identity of physical structures is grounded in the symmetry-groups of group theory which, in this sense, are prior to physical structures – they occupy a more fundamental position in the hierarchy of structures – and also to the physical objects physical structures are made up of.[95] The role that symmetry-groups play in WSS can be explained by appealing to the notion of *having a lower bound* introduced in chapter 3, defining a peculiar notion of fundamentality (weaker that that presupposed by Metaphysical Foundationalism). Even though quantum particles and quantum entanglement structures are not finitely grounded, they are bounded from below in *symmetry-groups*, which are not part of the grounding chain holding between quantum particles and physical structures – they belong to a mathematical, abstract level – and yet, as above, they are connected with physical structures by means of the notion of *shared structure.* On this basis, a weak notion of fundamentality can be applied to WSS, which I defined weak-fundamentality (W-F) in chapter 3. That being said, WSS turns out to have the main properties associated with WS and corresponds to the combination of mutuality (M), anti-reflexivity (AT), transitivity (T), and weak-fundamentality (W-F).

From this picture, the metaphysical commitments of WSS emerge; in particular, WSS entails a combination of the *Parity Thesis* (associated with *Moderate OSR*) and the *Priority Thesis* (endorsed by *Priority-based OSR*), where the latter is cashed out so as to include quantum entanglement structures and symmetry groups.

i) *Parity Thesis*: quantum particles and quantum entanglement structures are equi-fundamental.
ii) *Priority Thesis*: Symmetry-groups are prior to quantum entanglement structures and quantum particles in the sense of being their *lower bound*.

Before moving on, a reflection on the role that metaphysical explanation plays in WSS is needed. In sec. 4.2., I argued that one of the main benefits of adopting grounding in OSR is that this notion – in virtue of its close relationship with explanation – accounts for the structuralist idea that structures are *explanatorily* prior to objects. Of course, this way of articulating the explanatory connection between objects and structure should be reformulated in WSS, where physical structures and quantum particles belong to the same fundamental level. However, nothing prevents us introducing a broader notion of *holistic explanation* – taken to be even more powerful on some non-foundationalist views – which traces the relation of *Mutual Grounding* here proposed: structures ground and then explain objects for their identity and objects ground and then explain structures for their existence.

5. Weak Mathematical Structuralism (WMS) and Quasi-Thin Mathematical Objects

Chapter 5 returns to mathematical structuralism, illustrated in chapter 2, with the aim of examining more closely the metaphysical commitments of Shapiro's (1997) *ante rem* structuralism.

Provided with the metaphysical toolkit presented in chapter 3, I will first consider two main interpretations of Shapiro's account in terms of dependence (Linnebo, 2008) and grounding (Wigglesworth, 2018). Both approaches presuppose an orthodox view on metaphysical dependence, consistently with the standard metaphysical picture of Metaphysical Foundationalism (MF). Second, I will show that grounding is a more suitable candidate to make the relationship between objects and structures more precise and introduce Weak Mathematical Structuralism (WMS) as a further position in the mathematical structuralist debate. WMS will be developed in strict analogy with Weak Scientific Structuralism (WSS) in the philosophy of science. As I claimed in chapter 4, WSS allows us to avoid the 'relation without *relata*' objection to OSR, resulting from an entirely structural conception of physical objects. In a similar vein, WMS advocates a further strategy to escape the identity problem affecting Shapiro's *ante rem* structuralism and specifically its 'places-are-objects' perspective, where places are entirely reduced to their structural features. Indeed, the solutions to the identity problem that have been advanced may raise further objections on their own (chapter 2). The core idea is to put forward an alternative proposal by reconceptualizing the relationship between objects and structures and developing a variation of the 'places-are-objects' perspective, in which places are defined as more substantial *quasi-thin mathematical objects* – which are *something less* than objects occupying positions in systems but *something more* than Shapiro's (1997) mere positions. As I am going to show, WMS is worth endorsing because it avoids the identity problem without abandoning the main intuition of *ante rem* structuralism, that is the priority of structures. A more detailed introduction of WMS and its main features will be provided in section 5.2. In fact, before reconsidering the role that objects play in the structuralist ontology, we need to set the groundwork to articulate the standard structuralist ideas that ii) structures are *fundamental* and *prior to* objects and ii) objects are completely reducible to structures.

5.1. The metaphysical commitments of *ante rem* structuralism

As in scientific structuralism, the relationship between objects and structures in mathematical structuralism stands in need of further clarification. Not surprisingly, explicative and metaphysical notions such as dependence and grounding – which have been also invoked in scientific structuralism (chap. 3) – are up to this task, for they convey a distinctively non-causal priority relation among entities/facts. The appeal to metaphysical dependence in mathematical structuralism has two major advantages: first, it cashes out the distinction between mathematical platonism and *ante rem* structuralism – after all, both positions refer to abstract objects embedded in larger structures, but just the second deems objects less fundamental than or secondary to structures. Second, it makes some assumptions in *ante rem* structuralism more explicit. Consider, for example, the following passages in Shapiro:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (Shapiro, (2000, p. 353).

> The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it. (Shapiro, 1997, p. 78).

> If this notion of independence can be made out, we structuralists would reject it. The essence of a natural number is its relations to other natural numbers. (*ibid.*)

Despite not explicitly formulated in terms of metaphysical dependence, these claims clearly suggest that objects are somehow dependent on structures, in a fashion yet to be clarified. Of course, metaphysical dependence broadly understood comprises a variety of relations (see chapter 3, sec. 3.1.1 and 3.1.2). In the following sections, I will take into account Linnebo (2008) and Wigglesworth (2018)'s formulations of *ante rem* structuralism, which spell out the dependence under scrutiny in terms of essential and identity dependence (Fine, 1995; Lowe, 2005) and metaphysical grounding respectively. As I am going to show, both accounts successfully capture the metaphysical claims at hand in *ante rem* structuralism. In chapter 4 (sec. 4.1), I argued that *Priority-based OSR* – which appears comparable with *ante rem* structuralism (see chapter 2, sec. 2.2.4) – entails what I called the *Fundamentality Thesis* and the *Priority Thesis*. Arguably, these theses can be rephrased in *ante rem* structuralism in order to capture the fundamentality and priority claims at stake in Shapiro's account:

Fundamentality Thesis. All fundamental mathematical entities are structures.
Priority Thesis. Fundamental structures are prior to mathematical objects (if they exist).

However, grounding appears more suitable to secure the explanatory role that structures play in mathematical structuralism and to investigate the well-foundedness of the relationship between objects and structures.

5.1.1. *Ante rem* structuralism and dependence

Linnebo (2008, p. 59) begins by noting that «the notion of dependence figures prominently in many recent discussions of non-eliminative mathematical structuralism». Therefore, a more detailed analysis of this notion within mathematical structuralism deserves some serious attention. Linnebo focuses on *ante rem* structuralism, in which a discussion on dependence plays a crucial role, with two opposed views prevailing: that according to which all mathematical objects depend on structures and that which denies that such dependence relations obtain at all. Both approaches fail to provide an accurate and reliable interpretation of the metaphysical commitments of *ante rem* structuralism. For this reason, Linnebo aims at defending a more compelling compromise view, in which some mathematical objects depend on structures (e.g. abstract offices of the algebraic structures) but not others (e.g. sets).

Ante rem structuralism is committed to two main claims, which distinguish it properly from mathematical platonism: the *incompleteness claim* (mathematical objects are incomplete because they lack both an internal nature and non-structural properties) and the *dependence claim* (objects are dependent on the other objects in the same structure and on the structure they belong to). The former claim clearly recalls Parsons' (1990; 2008) characterization of objects in mathematical structuralism (see chapter 2, sec. 2.2.3) and includes two different assumptions:

1. Objects are incomplete if they do not possess intrinsic properties (I-incompleteness claim)
2. Objects are incomplete if they lack non-structural properties (NS-incompleteness claim).

To understand the first strain of the claim (I-incompleteness), Linnebo defines intrinsic properties as the properties that express the internal composition of objects, or the properties which an object would have «even if the rest of the universe were removed or disregarded» (pp. 65-66). So interpreted, I-incompleteness is promising, for it is true that mathematical objects cannot be considered in isolation from the structure they belong to. However, I-incompleteness collapses in the *dependence claim*: if objects lack intrinsic properties in the sense specified by Linnebo, i.e. they do not possess properties on their own, then they actually depend on the relevant structure.

The second interpretation (NS-incompleteness) presupposes a more precise characterization of structural properties. Linnebo (2008, p. 64) understands structural properties consistently with the so-called *invariance account* (see chapter 2, section 2.2.3):

> A structural property can now be characterized as a property that can be arrived at through this process of abstraction [Dedekind abstraction] or, equivalently, a property that is shared by every system that instantiates the structure in question.

When considering this interpretation, NS-incompleteness is subject to several counter-examples, concerning non-structural properties of objects:

> For instance, the number 8 has the property of being my favorite number. It also has the property of being the number of books on one of my shelves. It also has non-structural properties such as being abstract and being a natural number. In fact, the property of being abstract seems to be a very important property of natural numbers. (*ibid.*).

Considering these counter-examples, NS-incompleteness should be more modestly formulated, restricting structural properties of objects to a specific set of properties, i.e. *the properties that matter* *for their identity*; still, in doing so, we end up again with a formulation of the *dependence claim*: objects depend on structures for their identity. This motivates a stronger focus on dependence relations in *ante rem* structuralism.

Linnebo (2008) introduces two different dependence claims:

1) ODO (Objects Depend on Objects): each object depends for its identity upon all the other objects in the same structure.

2) ODS (Objects Depend on Structures): each object depends for its identity on the structure it belongs to.

(ODO) and (ODS) are not without problems; (ODO) has been criticized as a circular and not well-founded claim. The (ODS) claim, despite asymmetrical and then apparently non-circular, has been also put into question by Hellman (2001) and MacBride (2006), who have similarly pointed out that even though the identity of objects depends upon the relevant structures, the identity of structures presupposes *relata* having already been individuated or numerically distinguished.[96] Still, according to Linnebo, these objections are not conclusive and then they leave room for a more detailed analysis of dependence.[97]

Significantly, both claims lead to a compromise view, in which some – but not all – mathematical objects depend on the structures they belong to. Sets provide a counter-example to (ODO) and (ODS) because while sets depend upon their elements, the converse does not hold.

---

[96] In particular, Hellman (2001, pp. 193-194), by referring to positions in the abstract structure shared by systems satisfying the Peano-Dedekind axioms, claims that they «are entirely determined by the successor function [...] and derivative from it in the sense of being identified merely as the terms of the ordering included by [this successor function]. [...] but if the *relata* are not already given but depend for their very identity upon a given ordering, what content is there to talk of 'the ordering'?[...] This, I submit, is a vicious circularity: in a nutshell, to understand the *relata*, we must be given the relation, but to understand the relation, we must already have access to the *relata*». In a similar vein, MacBride (2006, p. 67) argues: «In order for objects to be eligible to serve as the terms of a [...] relation they must be independently constituted as numerically diverse. Speaking figuratively, they must be numerically diverse 'before' the relation can obtain; if they are not constituted independently of the obtaining of a [...]relation then there are simply no items available for the relation in question to obtain between».

[97] According to Linnebo, Hellman's (2001) objection involves a significant conflation between metaphysical and epistemic issues. If a metaphysical interpretation is suggested, then Hellman's argument relies on the following premise: «(RDO_1): the identity of any relation on a domain D presupposes that the individual objects from D have already and independently had their identities grounded» (Linnebo, 2008, p. 71). MacBride's (2006) objection, by contrast, stems from a slightly different premise, presented by Linnebo as follows: «(RDO_2): the obtaining of any relation presupposes that the objects it relates have already had their identities grounded» (*ibid.*). Either way, according to Linnebo, these premises are intuitive, but they are not properly justified.

According to the prevailing iterative conception, sets are 'formed from' their elements. The relation between a set and its elements is thus asymmetric, because the elements must be 'available' before the set can be formed, whereas the set need not be, and indeed cannot be, 'available' before its elements are formed. A set thus appears to depend on its elements in a way in which the elements do not depend on the set. (Linnebo, 2008, p.72).

At the same time, sets do not depend on the higher hierarchy of sets, so that an upward dependence does not hold at this level either: for example, the identity of a simple set such as a singleton depends only on its single element, «without even mentioning infinite sets» (p. 73) or the higher levels of the hierarchy of sets.[98]

By contrast, Linnebo refers to 'abstract offices' in algebraic structures obtained by a process of Dedekind abstraction – mapping a system, or a set of isomorphic systems, to its abstract structure – as an example in which (ODO) and (ODS) obtain. Linnebo begins by considering a system with a domain D and a set of relations $R_1, \dots, R_n$ on D. Take R to be the product relation of $R_1 \times \dots \times R_n$ and $\bar{R}$ to be the abstract structure of R, whose identity conditions are established by Dedekind abstraction on isomorphic systems: $\bar{R} = \bar{R} \leftrightarrow \mathrm{R} \cong \mathrm{R}$. Assuming an object x in R, the corresponding abstract office in $\bar{R}$ is denoted by $\tau(x, R)$. Abstract offices so defined are individuated as follows:
$\tau(x,R) = \tau(x',R') \leftrightarrow \exists f\,[f\colon R \cong R' \wedge f(x) = x']$.

Less formally,

The dependence claim appears to be true of structures obtained by this form of Dedekind abstraction. In particular, the abstract offices appear to depend on the structure to which they belong: each such office has its identity solely in virtue of belonging to this particular structure. (p. 76)

Once objects which are subject to dependence claims are identified, a more precise definition of the notion of dependence at hand is required. As mentioned above, objects depend on other objects and

[98] In fact, individuating sets in this way would involve a number of controversial issues which are not really required when dealing with very simple sets: «How far does the hierarchy extend? Are the different stages rich enough for the continuum hypothesis to fail? It would be a pity if very simple sets, such as the empty set and its singleton, depended on the entire hierarchy of sets, and their identities could therefore not be completely known before these hard questions had been answered» (Linnebo, 2008, p. 73).

on structures for their *identity*.[99] Linnebo specifically refers to Fine's (1994; 1995) essential dependence and Lowe's (2005) identity dependence (see chapter 3, sec. 3.1.1), which account in detail for the idea that «an object depends for its identity on another object if and only if any individuation of the former object must proceed via the latter» (Linnebo, 2008, p. 78).

Moreover, Linnebo (*ibid.*) distinguishes between a *strong* and a *weak* sense of dependence:

strong dependence: «*x* strongly depends on *y* just in case any individuation of *x* must proceed via y».

In accordance with Lowe (2003), *individuation* means the *explanation* of the identity of an object. Applied to sets, strong dependence entails that in order to individuate a set, its elements must be specified – it is impossible to individuate a set (e.g. the singleton of Socrates) without proceeding *via* the individuation of its elements (e.g. Socrates himself). The reference to Lowe's notion of individuation makes strong dependence more precise:

> Since it is essential to the singleton of Socrates that it is the value of the singleton function applied to Socrates as argument, this singleton depends on Socrates. But since it is not essential to Socrates that he is the value of the sole-element-of function applied to the singleton as argument, there is no dependency in the reverse direction. (Linnebo 2008, p. 78).

However, there is another, 'weak' sense of dependence which, according to Linnebo (2008, *ibid.*), «has received little or no attention» in the literature:

weak dependence: «*x* weakly depends on *y* just in case any individuation of *x* must make use of entities which also individuate y».

For example, a set *weakly* depends upon its subsets; this is because it strongly depends on its elements, which also suffice to individuate the set's subsets. Significantly, sets (which provide a counterexample to the dependence claim) do not *even weakly* depend upon their hierarchical structure of sets, whereas abstract offices (to which dependence applies) depend *only weakly* upon algebraic

[99] In fact, Shapiro (2008, p. 302) himself has rejected a form of modal-existential dependence (see chapter 3, sec. 3.1.1.) given that mathematical objects necessarily exist.

structures. In particular, the individuation of an abstract office proceeds *via* (i.e. strongly depends upon) an ordered pair (R, x), where R is a system that realizes a structure and x an element in this system. It follows that an abstract office weakly depends on the other offices and on the abstract structure itself, «for in order to individuate such an office we need a realization of the structure. But this is also all we need to individuate the relevant abstract structure itself». (Linnebo 2008, p. 79).

However, Wigglesworth (2018, p. 223) has objected that the proposed account of dependence turns out to be not available to *ante rem* structuralism – to which Linnebo implicitly restricts his metaphysical investigation; in fact, according to Linnebo's definition of strong dependence, the individuation of abstract structures must proceed *via* a realization R. Arguably, R does not refer to a *particular* system (for any other system exemplifying the relevant structure would suffice to individuate it); what it is required is that *some* systems realize such structure. Even if this is the case, it follows that abstract structures strongly depend on the existence of some systems exemplifying them; but this is the thesis that is actually rejected by *ante rem* structuralism and endorsed by *in re* structuralism.

5.1.2. *Ante rem* structuralism and grounding

Wigglesworth's (2018) interpretation of *ante rem* structuralism in terms of metaphysical grounding is supposed to supply a broader account, which relies on Linnebo's (2008) characterization of dependence and yet is consistent with an *ante rem* individuation of structures. The introduction of grounding relies on Linnebo's idea that the individuation of an object involves the explanation of its identity. Insofar as the explanation at hand is *metaphysical explanation*, the dependence claims also qualify as grounding claims – given the close relation between dependence, grounding and metaphysical explanation.

Before proceeding with the analysis of Wigglesworth (2018), let us sum up the main properties of grounding on the orthodox view of grounding, which are crucial to understand more deeply the application of this notion to structuralist claims. Despite the analogies between grounding and dependence, grounding is generally understood as a distinct metaphysical relation, which fulfills stricter conditions (anti-reflexivity, anti-symmetry and transitivity). At its core, grounding captures the idea that some things obtain *because* or in *virtue of* some other things. If dependence holds between *entities*, the *relata* of the grounding relation are either *entities* (on the *operational* account of grounding) or *fact*s or *propositions* (on the *relational* account). In particular, an entity/fact is said to be grounded in another entity/fact either for its *identity* or for its *existence*. In mathematical structuralism, grounding claims – exactly as dependence claims – plausibly involve the *identity* of entities/facts: Wigglesworth (2018, p. 225) specifically refers to facts: «the fact that one entity has the identity it has grounds the fact that another entity has the identity it has». Another important distinction is that between full and partial ground (see Fine, 2012). Full ground entails that x on its own fully grounds y; partial ground is generally defined in terms of full ground: x partially grounds y just in case there is something else together with x such that they jointly ground y.

In such grounding framework, *ante rem* structures are identified with *unlabelled graphs* (G) composed by nodes (n) and edges (E) between the nodes, corresponding to objects and relations respectively. Let be E_n the collection of structural relations that a node instantiates and **G** the isomorphism class of G.

With these clarifications at hand, let us investigate grounding claims in mathematical structuralism more deeply. In analogy with Linnebo's analysis (2008), two grounding claims are set out:

1) (ODO): for any mathematical objects, n_1 and, in the structure G, the fact that the identity of n_1 is E_{n1} partially grounds the fact that the identity of n_2 is E_{n2}.
2) (ODS): For any mathematical object, n, in the structure G, the fact that G ∈ **G** fully grounds the fact that the identity of n is E_n .

The comparison between mathematical structuralism and graph theory allows us to delineate identity criteria for structures: Wigglesworth argues that structures are not grounded for their identity in the nodes – which can be permuted leaving the graph unchanged – but in the operation of adding or removing an edge between the nodes, which would result in a different graph. This allows for an interpretation of grounding claims in terms of possible structures/graphs, which do not refer to any realization of the structure: «and so, unlike Linnebo's account, it is an account of grounding that is available to both the *ante rem* and *in re* non-eliminativist structuralist» (Wigglesworth 2018, p. 232). In a nutshell, the identity of a graph G is determined by its isomorphism class **G**. This is a standard definition of structures provided by Shapiro (1997, p. 93) in the context of *ante rem* structuralism:

> We stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one.

Hence, Wigglesworth's (2018) account of grounding has the advantage of preserving an *ante rem* individuation of structures.

A more detailed analysis of the properties of grounding provides further reasons to adopt grounding – rather than dependence – in order to account for (non-eliminative) structuralist claims. As shown in chapter 3 (sec. 3.1.3), even if both grounding and dependence involve metaphysical explanation, grounding is taken to have a stricter connection with metaphysical explanation, allowing in some cases for an identification of the two notions (i.e. in the *unionist* approach to grounding and explanation): in fact, grounding, by being anti-reflexive, admits a full overlap with explanation, which standardly entails anti-reflexivity.[100] This fits well with the structuralist idea that structures ground objects in the sense of metaphysically explaining their identity, and reinforces the priority of structures by securing their explanatory import in mathematical structuralism. In terms of metaphysical explanation, the identity of an object is *partially* explained by its relations with any other objects in the same structure (ODO) and *fully* explained by the structure – there is *nothing outside the structure* explaining its identity (ODS).

Another major advantage of interpreting *ante rem* structuralism in terms of grounding, to which I will come to in section 5.4., is summed up by Wigglesworth as follows:

> These cases of ground [mathematical objects are grounded in the structure they belong to and in the other objects in the same structure] are particularly interesting for the claim that the grounding relation is well-founded. If they are taken as genuine cases of ground, as we argue they should be, then they provide cases that involve infinitely descending chains of ground. These chains, however, are bounded from below. So they are non-well-founded in one sense, but well-founded in another. (Wigglesworth, 2018, p. 217).

[100] This is not the case for dependence, which can be reflexive (i.e. an entity ontologically depends on itself).

In other words, mathematical structuralism offers an interesting case-study for investigating the well-foudedness of grounding. Wigglesworth (2018, pp. 219-220) distinguishes three main ways in which grounding can be well-founded: i) *being finitely grounded* (to terminate in an un-grounded fact that occurs after a finite number of steps); ii) *being bounded from below* (each fact in a grounding chain is either fully grounded or identical to a fundamental fact *F* that does not need to be part of the relevant grounding chain); iii) *having a foundation* (a fact has a foundation in a set of facts *S* which are not grounded in the totality of facts).[101] (ODO) and (ODS) articulated in terms of grounding show that objects are *partially* grounded in the other objects of the same structure and *fully* grounded in the structure they belong to. On this view, grounding cannot be *finitely grounded* (i) in mathematical structuralism, for (ODO) brings about infinitely descending grounding chains involving the identity of objects within a structure. However, «[…]the identity of the structure itself is a full ground for each member of the chain, and so it is a lower bound of full ground» (Wigglesworth, 2018, p. 233). i) commits to ii) and ii) entails (iii), but the other way round does not hold; hence, if grounding claims are bounded from below, then they have a foundation as well, but they do not need to be finitely grounded.

For all these reasons, grounding seems to capture the relation between objects and structures more deeply, and it is also a better tool to formulate WMS, which is the objective of the following sections. Still, as I am going to explain, WMS relies on *Mutual Grounding*, in which the standard properties of grounding are deeply reconsidered consistently with a non-foundationalist perspective and – more precisely – with Weak Structuralism (WS) elaborated in chapter 3.

[101] *Being finitely grounded* and *being bounded from below* correspond to the two main senses in which fundamentality can be understood (i.e. *being well-founded* and *having a lower bound*) illustrated in chapter 3, sec. 3.1.

5.2. WMS and Mutual Grounding

The theoretical core of WMS consists of a significant reconsideration of Shapiro's interpretation of objects and the related identity problem. In *ante rem* structuralism, objects are entirely defined by their structural properties; when structures with non-trivial automorphisms are at hand, this conception meets with the serious issue of identifying objects which are mathematically distinct, suggesting that structures – in order to confer individuality on the *relata* – already *presuppose* objects of some sort. The distinction between Shapiro's (1997) entirely structural objects and my conception of quasi-thin objects will be spelled out by investigating their structural and non-structural properties.[102] While Shapiro's places as objects possess structural properties only, quasi-thin objects are endowed with both structural and non-structural properties. The analogy with Wolff's (2011) position in scientific structuralism – along with some relevant metaphysical considerations – will suggest a more precise interpretation of the non-structural properties of quasi-thin objects in terms of *kind properties*.

Weak Mathematical Structuralism (WMS) entails that mathematical objects and abstract structures stand in a new relationship of fundamentality, with a peculiar non-foundationalist flavor. In particular – in accordance with the relation of *Mutual Grounding* (if x $\text{grounds}_{(I)}$ y, then y $\text{grounds}_{(E)}$ x) elaborated in chapter 3 and applied to scientific structuralism in chapter 4 – objects and structures are mutually (not exactly symmetrically) grounded in each other.

In *Mutual Grounding*, two distinct claims hold at the same time: *Object Identity* (objects are grounded in structures for their identity) and *Structure Existence* (structures are grounded in objects for their existence). These claims will introduce two more specific definitions of numbers as quasi-thin objects, which will be useful to propose a possible solution to the identity problem. Alongside, I will provide a more specific interpretation of kind properties in the mathematical framework, observing that they turn out to be related to counting and measurement facts which highlight the different applicative uses of numbers. This interpretation is crucial to handle some non-trivial automorphism cases which are subject to the identity problem. I will particularly focus on the automorphism on a 2-nodes unlabelled and edgeless graph and on +1 and –1 in the relative number structure (with a possible suggestion of how to apply a similar strategy to +*i* and –*i* in the complex numbers struc-

[102] At the same time, It is important to distinguish my conception of quasi-thin objects, elaborated in a structuralist framework, from Linnebo's (2018) thin objects, based on Fregean abstraction principles and conceived of as entities which «do not make a substantial demand on the world» (Linnebo, 2018, p. 5).

ture). Apparently, this response to the identity problem raises a possible objection, concerning the problem of cross-structural identities in *ante rem* structuralism. In this context, I will show that three main routes can be taken, as pointed out by Shapiro (2006a); among them, I will opt for – and partially re-elaborate – Shapiro's proposal, which does not affect the tenability of my account.

However, WMS preserves some intuitions of *ante rem* structuralism, as showed by a third *Structure Identity* grounding claim: structures are grounded for their identity in their isomorphism types. In fact, structures will be individuated independently of the systems instantiating them, by referring to Shapiro's definition of structures. The identity of structures plays a twofold role: firstly, it retains the priority of structures, in accordance with an *ante rem* framework. Secondly, it provides a *bound from below* for *Mutual Grounding*, thus preserving WMS from typical objections of circularity.

5.2.1. Structural and quasi-thin mathematical objects

As I have distinguished objects in scientific OSR and *quasi-thin physical objects* (chapter 4, sec. 4.2.1) by focusing on their state-dependent and state-independent properties, the difference between Shapiro's places as objects and *quasi-thin mathematical object*s can be similarly investigated by considering their respective structural and non-structural properties.

Shapiro's (1997) places as objects are generally described as possessing structural properties only,[103] which determine their very identity: what they really are as opposed to all the other objects in the same structure. Recalling the incompleteness claim (Parsons, 1990; 2008), places as objects are incomplete in the two senses outlined by Linnebo (2008): (1) they lack intrinsic properties; (2) they lack non-structural properties. Assumption (1) adequately spells out the theoretical core of *ante rem* structuralism. In fact, understanding *places* in structures as endowed with intrinsic properties would be inconsistent with *ante rem* structuralism, and would rather commit to a a form of Platonism about them. In principle, *ante rem* structuralism is not inconsistent with Platonism about objects (one can be committed to a background ontology of self-standing structures and yet admit objects, i.e. the natural numbers, which possess intrinsic properties and exemplify a specific structure); ho-

[103] Shapiro (2006a; 2008) has more recently developed a more moderate position about objects, according to which they possess non-structural properties as well (the property of being abstract, non-spatio-temporal, of not entering in any causal relation, etc., 2006a, 116). Still, he has not really developed a view that accounts for mathematical objects in these terms.

wever, the same position appears quite odd if applied to Shapiro's places as objects which – by definition – have no more than their structural relations. Assumption (2), by contrast, is controversial in *ante rem* structuralism, for it is subject to the counter-examples mentioned by Linnebo (2008). Let us define the non-structural properties of objects more precisely:[104]

1) intentional properties (e.g. 'being my favorite number').

2) Applied properties (e.g. 'being the number of books on one of my shelves').

3) Metaphysical properties (e.g. 'being abstract').

4) Kind properties (e.g. 'being a natural number').

On this basis, it seems that a category of objects can be individuated, which I will call *quasi-thin mathematical objects*: such objects are incomplete just in the first sense delineated by Parsons: although they clearly lack intrinsic properties in any platonist sense, they are endowed with both structural and non-structural properties, as the following table shows.

Tab. 2.

Properties	**Structural Objects**	**Quasi-Thin Objects**
Intrinsic	☒	☒
Structural	☑	☑
Non-structural	☒	☑

I will draw the distinction between structural and non-structural properties as one between *essential* (structural) properties and *non-essential* (non-structural) properties; in fact, structural properties determine quasi-thin objects for their essential identity. Non-structural properties define them – if not

[104] The present distinction has been presented by Linnebo as part of the talk 'Pure Structure: One Over Many' (2016), *Foundations of Mathematical Structuralism*, MCMP (Munich Center for Mathematical Philosophy).

as individuals – as numerically distinguished *relata*, conceivable independently of the structure they belong to.

In order to make quasi-thin objects more precise, let us investigate the metaphysical features of non-structural properties 1- 4 illustrated by Linnebo (2008):

Tab.3.

Non-Structural Properties	**Essential**	**Necessary**	**Contingent**
Intentional	☒	☒	☑
Applied	☒	☒	☑
Metaphysical	☑?	☑	☒
Kind	☒	☑	☒

As outlined in table 2., intentional and applied properties such as 'being my favorite number' and 'being the number of books on one of my shelves' result in merely accidental properties, whose attribution to quasi-thin objects is too contingent to effectively distinguish them from (entirely) structural objects. Metaphysical properties such as 'being abstract' are more troublesome on a structuralist perspective: as acknowledged by Linnebo (2008), they are very important properties, actually appearing as non-structural but necessary (and plausibly *essential*).[105] Finally, kind properties such as 'being a natural number' are *necessary* properties of objects (i.e. the properties a number should have in order to be considered a natural, relative, rational number, etc.) though not properly *essential*:[106]on some conceptions of mathematical objects, the number 2, for example, is not *essentially*

[105] However, this issue is controversial in the literature.

[106] For a sharper distinction between essential and necessary properties see Nodelman and Zalta (2014), where essential properties are understood as *encoded properties* and necessary properties as *exemplified properties*.

natural, as it can be a natural as well as a relative, rational number, etc.[107] However, this idea is related to a broader issue – the cross-structural identity among places of different structures – to which I will come to in section 5.3.3.

On that view, kind properties are suitable for the present purposes: on the one hand, being non-essential, they fit well with an *ante rem* position; on the other hand, being necessary, they turn out to be important enough to elaborate quasi-thin objects in contrast with Shapiro's purely structural objects. Significantly, this interpretation has been already advanced by Wolff (2011) and McKenzie (forthcoming) in the context of scientific structuralism on quantum particles (see chapter 4, sec. 4.2.1), where state-independent properties such as mass and spin qualify quantum particles as different kinds of objects (electrons, muons, etc.). This idea seems to hold in mathematical structuralism as well, where kind properties are more promising than the other alternatives in order to establish the numerical diversity of objects. However, they are still in need of further clarification in the mathematical domain; I will suggest a possible interpretation in section 5.2.3.

Quasi-thin mathematical objects – similarly to *quasi-thin physical objects* – raise two main worries, which I will consider when addressing WMS:

i) are quasi-thin objects substantial enough to avoid resulting in a "no-objects-at-all" position?

ii) Are quasi-thin objects weak enough to preserve a structuralist framework?

5.2.2. Objects and structures: a new relationship of fundamentality

I will now delineate WMS starting from those formulations of *ante rem* structuralism which appeal to the notions of ontological dependence (Linnebo, 2008) and metaphysical grounding (Wigglesworth, 2018). Along the lines of Wigglesworth's (2018) proposal, WMS will be defined in terms of grounding, which fits better with the intention of applying a non-eliminative approach to both objects and structures. In particular, I will discuss the second grounding claim (i.e. ODS), that accounts for the relation between objects and structures. Although Wigglesworth understands this relation

107 Note that kind properties so understood are quite different from the interpretation of kind properties I suggested above in the context of OSR – where kind properties turned out to be secondary to structural properties and yet essential properties of quantum particles. In the mathematical framework, though, the interpretation of kind properties as essential is questionable, exactly because it depends on different interpretations of the cross-structural identities objection to *ante rem* structuralism.

as asymmetrical, in this context I advance an alternative interpretation, in which objects and structures are mutually – but not symmetrically – grounded in each other. This intuition can be expressed by the relation of *Mutual Grounding,* which is based on two *different* grounding claims holding at the same time – *Object Identity* and *Structure Existence*.

Object Identity: mathematical objects/nodes are fully grounded for their *identity* (not for their existence) in the structure/graphs they belong to.

Structure Existence: structures/graphs are fully grounded for their *existence* (not for their identity) in the existence of the objects/nodes constituting them.

Object Identity and *Structure Existence* involve the identity and the existential dimension respectively, thus motivating the *mutuality*, but not the proper *symmetry* of the relation.
A third *Structure Identity* grounding claim defines identity criteria for mathematical structures independently of objects themselves.

Structure Identity: structures/graphs are fully grounded for their identity in their own isomorphism types.

As I will show in section 5.4., *Structure Identity* will elucidate WMS's conception of extendability, which entails a peculiar interpretation of the relationship between *Mutual Grounding* and fundamentality.
Let us now cash out these claims one by one. I will firstly present the *Object Identity* claim, which implicitly refers to two assumptions:

Object Identity (a): objects/nodes are fully grounded in the relevant structure/graph for their *identity*.
Object Identity (b): objects/nodes are *not* fully grounded in the relevant structure/graph for their *existence*.

Consider an unlabeled graph with two nodes and no edges as a simple example:[108]

Fig. 1.

G': ○ ○

In the graph G', the nodes are interchangeable, because they can be permuted while leaving the graph unchanged; hence, their identity as individuals is solely determined by the graph they are in, as required by *Object Identity* (a). At the same time, *Objects Identity* (b) holds as well, since relations require *things* to stand in the relations, though such things are not objects in a proper sense. In the graph G, *Objects Identity* (b) is vindicated by the idea that the two nodes cannot collapse into one another, for they would result in a different (smaller) graph.

The automorphism defined on the graph G' can be easily compared with the permutation of quantum particles in singlet-state discussed in chapters 1. and 4.; exactly as it happens in the physical domain, neither node in G' can be said to be the node *a* and the node *b* – they are *structurally* indiscernible. Nevertheless, the way in which graph theorists use graphs in the mathematical practice suggests us that the two nodes are at least numerically distinguished, so that we are actually justified in defining the graph G' as a 'graph with exactly *two* nodes and no edges'. As I am going to show, much still needs do be done in order to ground their numerical diversity.[109] In section 5.3.1., I will propose a possible way to address this issue, by appealing to a specific interpretation of graphs and providing a more precise characterization of kind properties introduced in section 5.2.1.

Let us now consider the *Structure Existence* claim which, as *Object Identity*, includes two distinct claims:

---

[108] This is an interesting case, because it violates even weak PII (there is not a symmetric and irreflexive relation holding between the nodes which can weakly individuate them). By these means, Ladyman's (2005) solution to the identity problem (see chapter 2, sec. 2.4.1) is subject to a serious counterexample. This case has been illustrated by Ladyman and Leitgeb (2008) and Wigglesworth (2018) in order to argue for a primitive notion of identity as a possible solution to the identity problem. Still, this paradigmatic example is useful also in the present context, where a different proposal is put forward.

[109] To be more precise, the reference to the mathematical practice may suffice to distinguish the nodes in G' insofar they are given a primitive notion of identity (along the lines of Ladyman and Leitgeb, 2008). Still, my aim is to propose an alternative solution to the identity problem, which requires to account for the distinction between the nodes on different grounds.

Structure Existence (a): structures/graphs are fully grounded in individual objects for their *existence*.
Structure Existence (b): structures/graphs are *not* fully grounded in individual objects for their *identity*.

Concerning *Structure Existence* (a), the basic idea is that without distinct things existing metaphysically prior to the structure, there is nothing to stand in the relations that are supposed to confer individuality on the *relata*. At the same time, the identity of structures is to be settled independently of objects themselves, in accordance with *Structure Existence* (b).

This leads to the third *Structure Identity* claim:

Structure Identity: structures/graphs are fully grounded in their isomorphism types for their identity.

As acknowledged by Wigglesworth, the identity of structures does not depend on the very identity of objects – which can be permuted while leaving the graph unchanged – but on the operation of adding or removing an edge, that would result in a different graph. In a nutshell, the identity of graphs is determined by their isomorphism types, where no concrete systems are at play, consistently with Shapiro's (1997, p. 93) definition of structures.

WMS is intended to be a *metaphysical* position, which posits objects and structures in a new rela- tionship of fundamentality; Nonetheless, one could question the *epistemological* significance of this account, which is worth to briefly consider. In *ante rem* structuralism, the access to objects presupposes the access to the relations of the structures, thus suggesting the *epistemic* – not only the metaphysical – priority of structures. Leitgeb (2020, Part B, p.16), in developing a theory of *ante rem* structures as unlabeled graphs, favours an alternative approach, in which structures and objects are epistemologically on a par:

> No unlabeled graph without its nodes and edges, no nodes or edges in an unlabeled graph without the graph, and therefore one either understands all of them as a package or one does not understand any of them at all.

This proposal goes hand with hand with the metaphysical picture suggested by WMS and drafts a promising way of tackling the epistemological issues in a more moderate form of non-eliminative structuralism. Still, this is a topic for another work, whose aims go beyond the present discussion.

5.2.3. Numbers as quasi-thin mathematical objects

Object Identity and *Structure Existence* are consistent with the understanding of numbers as quasi-thin mathematical objects. In particular, two definitions can be outlined (in analogy with definitions 1.A and 2.A for quasi-thin physical objects, chapter. 4, sec. 4.2.3).

1. B. *Quasi-Thin Objects [Existence]*: Quasi-thin mathematical objects are things whose essential identity is grounded in the relevant structure, but whose existence is necessary to posit relations themselves.

2. B. *Quasi-Thin Objects [Kind]*: Quasi-thin mathematical objects are things that (in addition to their structural properties) possess also non-essential, non-structural kind properties (the properties that qualify numbers as natural, relative, rational, etc.) which suffice to fix their *numerical diversity*.

Definitions 1.B and 2.B allow elaborating numbers as *things* or *entities* being numerically distinguished and conceivable independently of the structures they belong to in virtue of their non-structural kind properties. However, a more precise description of kind properties in definition *Quasi-Thin Objects [Kind]* remains to be accomplished. As I mentioned above, some examples have been illustrated in the scientific framework, where Wolff (2011) refers to state-independent properties such as mass and spin to elucidate the kind properties of quantum particles. On the contrary, very few attempts to clarify kind properties in the mathematical domain have been proposed.

Intuitively, kind properties of numbers are strictly connected to their counting and measurement use in applicative situations. For this reason, they have been often confused with applied properties (cf. Linnebo, 2008). Consider the following quotation by MacBride (2005, p. 584):

> For what use are the cardinal numbers if they can't be employed to count? What merit is there in the real numbers if they cannot serve to measure? But while properties of application cannot be dismissed in this way, they cannot be reduced to the obtaining of structural relations either.

Actually, kind properties are not just related to items in the physical world that are used to count and measure. They also define the fundamental properties of each system of numbers at a more abstract level. For example, natural numbers (at least according to some conceptions, in which numbers are interpreted as *cardinals*, rather than as ordinals) respond to the question 'how many *Fs* there are?', whereas the rationals are defined for their role in measurement, i.e. as ratios between pairs of magnitudes. Kind properties so understood seem consistent with the metaphysical considerations put forward in section 5.2.1. On the one hand, by expressing the applicative *function* of numbers – and not their internal composition – they are *non-intrinsic* properties, as required by a structuralist account of objects. On the other hand, as acknowledged by MacBride (2005), they are *non-structural* properties – counting collections and measuring quantities can be seen as structure-independent operations.[110]

In the present context, I am referring to the natural, relative, rational, real and complex numbers as different kinds. Consequently, the natural numbers structure, the relative numbers structure, etc. can be understood as different structures. Given the present understanding of kinds, one could wonder if *any* arbitrary structure constitutes a kind. Recall that a structure is defined by a domain D with a specific cardinality and an order relation (or a set of relations) R, so that we obtain S = <D, R>. As an example, consider the natural numbers structure $\mathbb{N}$, with D = {0, 1, 2, 3,...} and R = *s* (i.e. the successor relation). However, we can also take into account less standard structures, in which either the cardinality of the domain D or the order relation R are modified. As an example of the first case, consider a finite substructure of $\mathbb{N}$, which I call $\mathbb{N}$', with D = {0, 1, 2} and R = *s*. In this case, there is variation on the cardinality of D such that D includes a collection with just three places. Alternatively, we could also take a structure which preserves the domain of $\mathbb{N}$ and yet assumes, in place of *s*, a different order relation R: for example, one in which 0 is not the initial element of the structure, but the successor of another natural number-place, so that we obtain a circular

---

[110] Again, this presupposes the understanding of numbers as cardinals, and one in which cardinals are not derived from ordinals.

structure or a loop. Let us define this structure ℕ''. We can also vary the order relation in a different structure: think for instance of a structure which has the domain of the real numbers structure but a different order relation, so that the interval between, say, 1 and 2 is discrete and not continuum. Let us label this structure ℝ'.

Do ℕ', ℕ'' and ℝ' correspond to specific kinds, distinguished from the natural and real numbers kinds and displaying peculiar kind properties? Intuitively, we do not have reasons to rule out this possibility. So far, I argued that kind properties correspond to applicative properties. If this is the case, it is not implausible to conceive the kind properties corresponding to ℕ', ℕ'' and ℝ' and then to understand them as different kinds; in fact, the applicative properties of these structures are distinct from those of ℕ and ℝ, for they apply either to a finite collection of objets or to collections in which the order relation has been significantly changed. Of course, this would commit to an abundant conception of properties and to a sort of ontological inflation.[111] Still, it is worth noting that such kind properties do not appear significant for the mathematical practice, and this motivates the present focus on the naturals, relatives, rationals, etc. as the *fundamental kinds,* whose applications are relevant and actually at use in mathematics. On this basis, non-standard structures such as ℕ', ℕ'' and ℝ', despite corresponding to existent and specific kind properties, can be plausibly left aside for the purposes of the present discussion.

The question concerning whether graphs (and specifically the unlabelled graph with two nodes and no edges) give raise to kinds is a different one. Some reasons to interpret graphs as specific kinds – and *significant* ones, unlike the aforementioned ℕ', ℕ'' and ℝ' structures – will be suggested below, by highlighting a feature that would distinguish them from other mathematical entities and, possibly, also from the fundamental numbers kinds discussed above.

[111] See Lewis (1986) for the distinction between an abundant (maximalist) and a sparse (minimalist) conception of properties.

5.3. A possible response to the identity problem

WMS and quasi-thin mathematical objects as defined in sections 5.2.2. and 5.2.3. introduce an alternative strategy to avoid to the identity problem concerning structures with non-trivial automorphisms. The proposed solution is based on the possibility of attributing to objects in non-rigid structures, which share the totality of their structural properties, different non-structural *kind properties.* As for Weak Scientific Structuralism (WSS), the opposition between *entities* and *objects* – in which entities are the subject of a predication of properties and objects are properly individuated entities – plays a crucial role: consistently with a structuralist framework, I maintain that mathematical objects should be *individuated* by their structural, essential properties. Therefore, objects in non-rigid structures remain *structurally* indiscernible. Still, I argue that such objects are at least *numerically distinct* in virtue of their kind properties, which allow us to distinguish them as more generic *things* or *entities.*[112] In fact, kind properties, as applicative properties strongly dependent on *context*, shed light on how different kinds of mathematical objects are actually used in the mathematical practice. On this basis, even though the relevant entities cannot be properly individuated – after all, we cannot establish which one is which, as pointed out by Burgess (1999) and Keränen (2001) – they cannot collapse into one another either, for they are often used or applied in very different ways.

Of course, this general strategy needs to be drawn up with respect to specific cases of non-trivial automorphisms. In what follows, I will focus on the case of an unlabelled graph with two nodes and no edges and the case of +1 and –1 in the relative numbers structure.

5.3.1. The case of graphs

The unlabeled 2-elements graph G' presented so far provides a simple case of non-trivial automorphism, where the nodes in question prove to be structurally indiscernible as *objects* but numerically distinguished as *things* or *entities*. In order to spell out their numerical diversity, let us start by evaluating which kind-applicative properties graphs might have. Standardly, graphs are mathematical entities; however, graphs have the function of providing figurative representations of the relations among objects in the physical world. Unlike other mathematical entities, graphs have a strong diagrammatical and schematic aspect, which makes graph-theoretic representations *fundamental* to ex-

[112] For the distinction between numerical diversity and identity, see chapter 4, sec. 4.2.2.

plain the relations in physical systems. On this view, graphs are strictly connected with their concrete counter-parts,[113] thus displaying the kind property of being applicable to physical systems. Let us define this property K-APP. This would distinguish them from other abstract mathematical entities, which do not leave room for such powerful empirical applications.[114] The connection between graphs and their concrete representations has been specified, among others, by Pincock (2007): in order to support the theoretical indispensability of mathematics for physics, Pincock tackles the well-known problem of the seven Königsberg bridges by appealing to graph theory, which gives a compelling mathematical explanation of why it is impossible to complete a 'Eulerian path'[115] (crossing all the Königsberg bridges passing only once on any single bridge and returning to the starting point). Pincock (2007, p. 259) advances the following interpretation of the relation between bridges and the graph abstractly representing them:

> It is tempting to say that the bridge system just is a graph, although this is somewhat misleading. The bridge system is of course not a graph because graphs are mathematical entities and the bridge system is physical. Still, the bridge system and this particular graph seem much more intimately connected than the system with a temperature and the number 40. We might capture this by saying that the bridge system has the structure of a graph, in the sense that the relations among its parts allow us to map those parts directly onto a particular graph.

K-APP seems consistent with the characterization of kind properties previously provided for numbers. First, K-APP is not intrinsic: as for numbers, it sets out an applicative function of graphs, and not an inherent feature of them – as pointed out by Pincock, graphs are mathematical entities and they should not be identified with bridges, which are just (one of) their concrete representations.

---

[113] This idea echoes Parsons'(2008) conception of quasi-concrete objects: «objects of a kind which goes with an intrinsic, concrete 'representation', such that different objects of the kind in question are distinguishable by having different representations» (Parsons, 2008, p. 34). Still, what I am here proposing is a slightly different approach, in which graphs are abstract entities and yet they are closely related to physical systems (recalling Pincock, 2007).

[114] For example, even if we maintain that graphs are abstract objects, they can be contrasted with what Parsons (2008, p. 36) defines 'pure abstract objects': natural numbers (and this plausibly extends to the other numbers kinds) and pure sets. With respect to natural numbers, Parsons (*ibid.*) claims: «For a particular number, say the number five, what could be meant by an intrinsic concrete representation of it? A possible answer would be: Any configuration of five objects. However, something is concealed by the word "configuration." If by a configuration is really meant something concrete and spatial, then it does not determine the kind of object such that it consists of five of them».

[115] As specified by Pincock (2007, p. 257) «A *path* of a graph is a series of edges where one of the vertices in the *n*th edge overlaps with one of the vertices in the *n* + 1th edge»

The question concerning whether K-APP is a structural property is more controversial. On the one hand, the graph-theoretic figurative representation preserves the *structure* of the concrete system it is linked to, mapping its relations by abstraction. On the other, this is quite independent from the original graph itself: to be more precise, K-APP is not a structural property in the same sense in which, for example, 'being the successor of' is a structural property in *ante rem* structuralism. While the latter is a structural property 'internal' to the structure, the former just points to an *empirical application* of the graph which works by a structural mapping. On this basis, I would define K-APP as a *mapping* property, establishing a specific relation between a mathematical entity (in this case, a graph) and an empirical system. This may suffice to distinguish K-APP from Shapiro's understanding of structural properties – which by contrast are intended to apply *within* a mathematical structure.

That being said, let us apply the automorphism on the two nodes in G' to an empirical situation. Consider a variation of the Königsberg bridges example referring to the current number of Königsberg bridges, which have been reduced from seven to five. In this new arrangement, take the two nodes in G to be two arbitrary points in two physically *identical* islands among which, say, the sixth bridge – now removed – used to be. This scenario offers a concrete representation of the 2-elements unlabelled graph G with no edges: we have two indiscernible points, for we still do not know which one is which. The two points are apparently identical and share the totality of their qualitative properties. However, we can numerically distinguish them by means of their spatio-temporal location, since we know that the two points occupy two different positions in spacetime.[116] So, if we assume that the kind properties of graphs are those of being applicable to physical systems, then the two nodes in G' are distinguishable in virtue of their kind properties K-APP i.e. they have different empirical applications, consisting of the two identical points differently located in spacetime.

[116] I am not here addressing the issue concerning whether spatio-temporal location qualifies as a structural or a non-structural property, for this would involve the wider debate on substantialist *vs* relationalist approaches on space-time.

5.3.2. The tale of +1 and –1

Let us now investigate more complex cases of non-trivial automorphism: consider, for instance, +1 and –1 in the relative number structure, in which +1 belongs to the positive *subset* (0, 1, 2, . . .) of the relative numbers (0, 1, 2, . . .; –1, –2, . . .). Arguably, the subset (0, 1, 2, . . .) *corresponds to* the natural numbers kind. So, if taken on its own, +1 displays a set of kind properties that is plausibly distinct from that of its negative counterpart –1 (consisting of the properties of the positive integers plus those needed in order to subtract any two numbers in ℕ).[117] Differently put, the properties of the positive integer +1 in $\mathbb{Z}^+$ overlap up to a point with the properties of the natural number 1, which I define kind properties (+)ℕ, i.e. the properties of operating with positive quantities, stemming from natural numbers being closed under addition. In fact, it is just when we consider the negative integer –1 that we are able to introduce the kind properties of relative numbers as a different set of properties that – extended from (+)ℕ – we can label (±) ℤ: the properties of making operations not only with positive quantities, but also with negative ones, for relative numbers are closed under both addition and subtraction. On this basis, I argue that +1 and –1 in ℤ are discernible in virtue of their different (non-structural) kind properties. This solution has the advantage of not involving either a primitive notion of identity or the reference to a weak form of PII,[118] thus presenting a third-way strategy to overcome the identity problem.

This proposal is not without problems and it is questionable whether it can be applied to other cases of non-trivial automorphisms. Let us then evaluate if a similar solution can be extended to the automorphism on +*i* and –*i* in ℂ. Recall that complex numbers (a + bi) resolve equations which are impossible in the real numbers structure: in particular, equations of the form $x^2 = -1$, where x is a free variable and the square of a real number is a negative number. The problem with complex numbers is that any formula (φ)x, with only x free, holds for both (a + bi) and its conjugate (a – bi), so that the two numbers are indiscernible. This leads to identify +*i* and –*i*, contradicting the claim that apart from 0, each number has two *distinct* square roots. In this context, the following quotation by Shapiro (2006a, p. 139) may be helpful:

[117] When it comes to kind properties, I assume that the positive integer +1 can be considered in isolation since kind properties are non-structural properties

[118] See chapter 2, sec. 2.4.

> With complex analysis, we can distinguish i from $-i$ as follows: the pair $\langle i, -i\rangle$ satisfies the formula $x + y = 0$, and the pair $\langle i, i\rangle$ does not. If i were identical to $-i$, then the ordered pair $\langle i, -i\rangle$ would be the same as the pair $\langle i, i\rangle$, and so these pairs would satisfy the same formulas. Of course, we still have no way of telling, among i and $-i$, which is which, but maybe we do not need that ability.

Shapiro acknowledges that Keränen (2001, p. 324) rules out the possibility of solving the identity problem by appealing to formulas with more than one free variable, because of the equivalence of isomorphic systems. However, Shapiro (2006a, p. 140) also notes that Keränen refers to the *individuation* of objects, not just to the possibility of numerically *distinguishing* them, which is not similarly undermined by the isomorphic equivalence. As Shapiro, I maintain that this is all we need to come up with a possible solution of the identity problem;[119] after all, on my account of WMS and quasi-thin objects, numbers are grounded for their identity in the structure they belong to, whereas structures are grounded for their existence in distinct *relata* (which I labelled quasi-thin objects). As I argued, quasi-thin objects can be at least numerically distinguished in virtue of their non-structural kind properties. Therefore, Shapiro's claim has the benefit of elucidating which kind properties complex numbers appear to have and then can be accommodated in my own WMS. More specifically, $+i$ extends the kind properties of real numbers in order to solve $x^2 = -1$ equations, where x is the only free variable; arguably, the kind properties of $+i$ are distinct from those of its negative conjugate $-i$, which further extend the properties of real numbers in order to solve equations not only with one free variable, but also with two free variables. Clearly, this is just a sketch of a proposal and much more should be said. Still, this is intended to be a way of gesturing to a possible application of the strategy proposed for +1 and –1 in $\mathbb{Z}$ to $+i$ and $-i$ in $\mathbb{C}$, by focusing on a possible account of their kind properties.

That being said, quasi-thin mathematical objects as defined by *Object Identity* (a; b) and *Structure Existence* (a; b) appear to be *substantial enough* to provide a possible response to the identity problem, addressing the first issue introduced so far (i. are thin objects substantial enough?). Unfortunately, the proposed strategy (especially as regards the case of +1 and –1 in $\mathbb{Z}$) meets with another standard difficulty of *ante-rem* structuralism, i.e. the problem of cross-structural identities, to which the next section is devoted.

---

[119] Significantly, differently from the strategy here suggested, Shapiro (2006, p. 139) ends up supporting a primitive notion of identity for $+ i$ and $-i$: «if the identity sign is a primitive of the language, and we have singular terms denoting each object, then we can trivially individuate each object. In complex analysis, i satisfies $x = i$ and nothing else does.»

5.3.3. Cross-structural identities: a further concern

The identity problem discussed so far concerns the identity of places in the same structure. A similar problem arises when it comes to places in different structures: is the natural number 2 identical to the real number 2?[120] This issue has an intimate connection with the solution to the identity problem I proposed above; in the case of the relative numbers structure, I argued that +1 is distinguished from its negative counterpart –1 because of its kind properties, which correspond to those of natural numbers. On that view, it is quite natural to ask whether +1 in the relative number structure should be identified with 1 in the natural numbers structure. In what follows, I will specifically target this case, that is particularly relevant for the present discussion.

Shapiro (2006a) claims that three main pathways have been explored in the literature, and that three main interpretations of the indeterminacy under scrutiny are available:

1. first, there is no fact of the matter on this issue; the numeral for the natural number 1 and the numeral for the relative number 1 have *determinate references* – in the natural number structure and in the relative number structure respectively. However, one is not compelled to take a stand on the identity or diversity of these referents, as the «the indeterminacy is charged to the (mathematical) world, not to mathematical language. It is an ontological or a metaphysical indeterminacy» (Shapiro, 2006a, p. 127).

2. Second, there could be a determinate identity relation between places of different structures – the natural 1 is identical to the integer 1 – but some of these identity relations remain inscrutable, and then subject to an *epistemic indeterminacy*.

3. Third, one could hold that places in different structures are distinct; according to Shapiro (2006, p. 128) this option fits better with one of the core intuitive ideas of *ante rem* structuralism, that is that objects are tied to the structures constituting them: if numbers are defined by their place in a specific structure, then objects in different structures are to be distinct. On that option, the relevant indeterminacy is a *semantic* one, concerning the mathematical language:

[120] This problem echoes the one due to Benaceraff (1965): «according to von Neumann's interpretation of arithmetic, each natural number is the corresponding finite ordinal, so that 2 is {φ, {φ}}; according to Zermelo's interpretation, is{{φ}}. So which is 2?» (Shapiro, 2006, p. 122).

> On this view, the lingering indeterminacy is charged to the language. The phrase 'the number 2' is systematically ambiguous, denoting a place in the natural number structure, a place in the integer structure, a place in the real number structure, a place in the ordinal structure, etc. (Shapiro, 2006a, p. 128).

This is the route taken by Shapiro (2006a): even though the mathematical language leaves largely un-explained which objects and which structures we are referring to, such indeterminacy does not interfere either with the mathematical practice or with matters of understanding and communication. In fact, as far as the relevant mathematical systems are isomorphic to each other, the mathematician is allowed to go back and forth a structure and any of the systems exemplifying it.

These three ways have pros and cons; let us then evaluate them in light of the kind properties illustrated so far, which play a crucial role in my own solution to the identity problem.

To begin with, the option (1) is apparently the least demanding from a philosophical point of view; after all, it does not make a difference whether places from different structures are identical or distinct, for this is a mathematical indeterminacy. As a mathematical issue, the problem of cross-structural identity does not affect either *ante rem* structuralism or – intuitively – WMS, since they are both philosophical positions. However, option (1) seems to pass the buck to mathematical practice despite having significant philosophical consequences, and not desirable ones: in particular, this option runs counter the Quinean slogan 'no entity without identity', according to which entities in a given theory should be given definite identity criteria.

Let us then focus on options (2) and (3), for which the questions about cross-structural identities make sense at a philosophical level. Option (2) turns out to be the most troublesome for my account. Identifying objects from different structures – although some of these identity relations remain unknown – would undermine the very idea of kind properties as distinguishing the naturals, the relatives, the reals, etc., thus threatening seriously the whole proposal of quasi-thin mathematical objects as defined by *Quasi-Thin Objects [Existence]*(1.B) and *Quasi-Thin Objects [Kind]* (2.B). At a first glance, quasi- thin mathematical objects are best vindicated by option (3), where the natural and the relatives are distinct and therefore have different kind properties. Still, in this framework, kind properties such as 'being a natural number' are allegedly *structural* and *essential* – what distinguishes the naturals, the reals, etc. are the specific structural relations in which they stand. This contradicts the definition of kind properties as necessary and yet *non-structural* and *non-essential* properties of numbers (cf. sec. 5.2.1). So, also option (3) appears to have unpleasant implications for WMS and quasi-thin mathematical objects as a possible solution the identity problem.

However, I believe that option (3) can be accommodated in WMS insofar as kind properties are more straightforwardly defined. What exactly does 'to be a natural, relative, etc. number' amount to? As I argued in section 5.2.3, a plausible explanation of kind properties is to identify them with *applicative properties* (cf. MacBride, 2005) involved in counting collections and measuring quantities in pragmatic situations. According to MacBride (p. 584), the way in which numbers are *used* cannot be reduced to the «obtaining of the structural relations». On this view, kind properties of numbers can be seen as non-structural properties, consistently with the definition provided above in the framework of WMS. If we interpret kind properties as applicative properties, then a stronger focus on *context* is required – a *contextual indeterminacy* emerges along with a semantical one. However, this is not always the case: in some applications, the reference to a particular context can shed light on which numbers we are actually using, despite their being semantically indeterminate. Some examples, involving different mathematical operations, will make this more precise. To begin with, consider an operation on natural numbers, such as $2 + 1 = 3$: this appears to be ambiguous both *semantically* and *contextually*: at the semantical level, 2, 1 and 3 denote naturals but also relatives, rational, reals etc. Consequently, at a contextual level, we cannot infer their applicative uses univocally, for it is not clear from the context which numbers we are exactly talking about.

Let us now turn to the relative number structure, that is relevant for the present discussion, and take into account as a possible example $1 + (-4) = -3$. On the one hand, a semantical and a contextual indeterminacy arise; upwardly, the same operation also semantically refers to 1, -4 and -3 in the larger structures of the rational and the real numbers, in which the relatives are structurally embedded. As in the previous case, the context does not allow us to distinguish their applicative uses either. On the other hand, $1 + (-4) = -3$ cannot be indeterminate if applied downwardly, i.e. to the substructure of the natural numbers, for -4 is not part of the naturals. Therefore, even though *semantically* 1 denotes the same place in both $\mathbb{Z}$ and $\mathbb{N}$, the applicative context in which they are considered helps distinguishing them, without necessarily appealing to their structural relations. Such reasoning can be easily generalized to other systems of numbers; for example, if we take into account the operation $\frac{2}{7} + \frac{4}{7} = \frac{6}{7}$ on rational numbers, a semantical indeterminacy arises upwardly: the same operation refers to $\frac{2}{7}$, $\frac{4}{7}$ and $\frac{6}{7}$ in the larger structures of the real and complex numbers.

However, this is not the case downwardly, for $\frac{2}{7}, \frac{4}{7}, \frac{6}{7}$ and are not part of the relatives. It is worth noting that the case of complex numbers is slightly different; if we focus on the following operation on complex numbers $(3+2i) + (5-4i) = (8-2i)$, we can more explicitly infer from the context that we are referring to complex numbers. In fact, complex numbers do not have clear correspondents in other systems of numbers, either upwardly ($\mathbb{C}$ is the largest structure of numbers) or downwardly (complex numbers include i and $-i$ as specific constituents).[121]

In other words, the applicative uses of the natural, relative, rational, real and complex numbers remain distinct whether they are treated as identical or distinct in ontological, epistemic or even semantical terms.[122] On this basis, I argue that the possible solution to the identity problem advanced in section 5.3.2 does not commit one to identify the relative +1 and the natural 1 – something that would be in contrast with both *ante rem* structuralism and my own account of WMS. In fact, along the lines of Shapiro's (2006a) option (3), I hold that places from different structures are actually distinct, and that their indeterminacy is charged to the language, not to the mathematical world. Nevertheless, I assume that +1 in $\mathbb{Z}$ and 1 in $\mathbb{N}$ are distinct not because of their structural properties – as Shapiro (2006a) puts it – but, rather, because of the applicative *context* in which they are used, which helps defining them as distinct in a non-structural way.

To better understand how the identity problem (a) and the cross-structural identities problem (b) are related in my proposal, let us come back to the automorphism on +1 and –1 in the relative numbers structure, which are clearly subject to both (a) and (b).

a): In section 5.3.2, I argued that +1 in $\mathbb{Z}^+$ corresponds up to a point with the natural number 1 in $\mathbb{N}$, enabling us to operate on collections of positive quantities only. In this sense, the integer +1 in $\mathbb{Z}^+$ shares with natural numbers the kind properties I defined $(+)_{\mathbb{N}}$, i.e. that of being added to any natural

[121] Of course, things are more complicated than that. For example, take the complex number a+bi, where the real part a is 1 and the imaginary part b is 0, so that we obtain $1+0i = 1$. In this case, someone could identify the result of the equation '1' with the real number 1. Still, the diversity between the complex 1 and the real 1 can be established by pointing out that that the equation at hand relies on a different process which – involving the imaginary unit i – results in 1 as a complex number. Moreover, the complex field allows for applications which are not possible with real analysis (as an illustrative example, take the application of complex numbers to solve physical problems, such as those in the electromagnetic field).

[122] Given the importance of the 2-elements unlabelled graph G in the present discussion, let us briefly consider whether graphs raise a similar cross-structural identity problem. In this respect, Leitgeb (2008, part B, p. 8) distinguishes between graphs and the numbers structures: «In the context of unlabeled graph theory, this is not much of an issue, as graph theorists do not seem to identify vertices across distinct unlabeled graphs anyway, differently from what happens in the mathematical practice, where it is common to identify naturals, integers etc.».

number by closure. By contrast, the integer –1 provides us with an additional ability, i.e. that of subtracting it to any natural number by closure, so that we can operate on collections of both positive and negative quantities. Consequently, –1 displays a different set of kind properties, extended from those of natural numbers in order to count collections in which negative quantities come into play (the counting of collections with just one/two…individual(s)) which I labelled $(\pm)_{\mathbb{Z}}$.

(b): At the same time, the reference to mathematical operations offers a broader contextual framework which allows us to discriminate between +1 in $\mathbb{Z}$ and 1 in $\mathbb{N}$; that is because the former can be used in larger range of mathematical operations on both positive and negative numbers (such as '1 + (– 4) = – 3') whereas the latter cannot.

On this basis, I assume that the integer +1 corresponds to the natural 1 (and this suffices to distinguish it from the integer –1) without being *identical* to it (so that we are not committed to the cross-structural identity of the natural 1 and the integer 1). That is because +1 in the relative numbers structure possesses both the property $(+)_{\mathbb{N}}$ (when it is considered within $\mathbb{Z}^{+}$) and the property $(\pm)_{\mathbb{Z}}$ (insofar it is embedded in $\mathbb{Z}$ on the whole).

Therefore, I believe that the problem of cross-structural identities does not directly challenge WMS and quasi-thin objects as presented in sections 5.2.2. and 5.2.3. Let us then proceed in the next section with the analysis of WMS, examining the individuation of abstract structures in more detail.

5.3.4. Tacking Stock: WMS as a middle-ground approach

Object Identity and *Structure Existence* have been useful to introduce quasi-thin mathematical objects in the structural ontology. However, quasi-thin objects do not commit to a form of *in re* structuralism, according to which abstract structures depend on the systems instantiating them. Conversely, they can be framed in an *ante rem* individuation of structures – where no systems are at play – as showed by the *Structure Identity* grounding claim, which grounds the identity of structures in their isomorphism types. By these means, the priority of structures is preserved, as required by non-eliminative structuralism. Wigglesworth (2018, p. 233) states that the individuation of structures *via* their isomorphism classes serves a *bound from below* for the grounding chains holding in mathematical structuralism: as suggested in section 5.1.2, the idea of a lower bound accounts for a non-standard interpretation of the well-foundedness of grounding in a structuralist framework – one in which there is not a finite number of steps between each element of the grounding chain and the *fondamentalium* that grounds it.

Significantly, this conception can be applied to WMS as well; on the one hand, *Object Identity* and *Structure Existence* ensure that objects and structures are on a par, and then not finitely grounded. In fact, although WMS includes two different grounding relations, their *mutuality* within the overall picture ensures that objects and structures belong to the same fundamental level, in contrast with the standard structuralist idea that structures only are *fundamental*; hence, the resulting grounding relation appears to be not well-founded in the standard sense (i.e. finitely grounded). On the other hand, the identity of the structures is a *bound from below* (or a full ground) for both claims, thus providing WMS with a (non-standard) foundation, which (in accordance with the metaphysical properties of WS described in chapter 3) I define weak-fundamentality (W-F). On these grounds, quasi-thin mathematical objects – which turned out to be *substantial enough* to avoid the identity problem – also respond to the second issue introduced so far (ii. are quasi-thin objects weak enough?), because they appear *weak enough* to retain a non-eliminative *ante rem* approach – where abstract structures are ultimately prior to the objects composing them.

These clarifications make it plausible to say that WS involves M, AR, T, W-F, where M and W-F are intended to be variations of anti-symmetry (¬AS) and fundamentality (F) respectively and anti-reflexivity (AR) – as opposed to Coherentism – can be endorsed again. Indeed, each direction of grounding that I have taken into account is anti-symmetrical on its own, thus not leading back to the starting point, as in coherentist circles of ground. As I argued for Weak Scientific Structuralism (WSS), the combination of properties of WMS allows us to infer more precisely its metaphysical commitments – as opposed to the metaphysical claims at hand in *ante rem* structuralism – which are captured by the following theses (re-adapted from theses i-ii attributed to WSS):

iii) *Parity Thesis*: mathematical objects and mathematical structures are equi-fundamental.

iv) *Priority Thesis*: isomorphism types are prior to mathematical structures and to mathematical objects in the sense of being their *lower bound*.

Hence, WMS results in a mutual, but not exactly symmetrical position, so as to avoid typical circularity objections concerning not well-founded structuralist ontologies. Moreover, a form of holistic explanation holds in WMS as well, going together with the relation of *Mutual Grounding* proposed; objects and structures are explained by each other, but structures explain objects for their identity and objects explain structures for their existence.

Concluding Remarks: Towards a New Taxonomy of Reality

Scientific Ontic Structural Realism (OSR) and mathematical *ante rem s*tructuralism are intimately related positions, which assume structures to be *fundamental* and *ontologically prior* and objects to be entirely reduced to their structural features. Both views are motivated by specific problems in scientific and mathematical structuralism. OSR aims at overcoming the metaphysical under-determination problem affecting quantum particles in QM – which are consistent with two metaphysical packages, i.e. quantum particles as individuals and as non-individuals – by replacing the standard object-oriented metaphysics with a structural ontology, in which objects are either eliminable or reducible to structures. Shapiro's *ante rem* structuralism, by contrast, is meant to combine realism in ontology and realism in semantics with an acceptable epistemology. To this aim, Shapiro (1997) introduces a full-fledged theory of structures in which objects are just places or positions within them, in accordance with the 'places-are-objects' perspective.

In both cases, all that matters about objects are their *structural properties*, an assumption that has been deeply challenged in both theoretical frameworks. OSR has elicited the 'relation without *relata*' objection (Cao, 2003; Dorato, 1999; Psillos 2001, 2006; Busch, 2003; Morganti 2004; Chakravartty, 1998; 2003), claiming that OSR collapses into absurdity by eliminating the *relata* which should make up the relations and hence the structures OSR is concerned with. Mathematical *ante rem* structuralism also ends up in a counter-intuitive result, committing to identify objects (i.e. those belonging to structures with non-trivial automorphisms) which are actually distinct in mathematical practice. This introduces the so-called *identity problem* for Shapiro's (1997) 'places-are-objects' perspective (Burgess, 1999; Keränen, 2001) and related difficulties concerning the interpretation of the Principle of Identity of Indiscernibles (PII) in mathematical structuralism. The existing solutions to these problems are not entirely convincing.

In the debate on OSR, some more defensible, non-eliminative positions have been proposed, i.e. *Priority-based OSR* and *Moderate OSR*. However, these views – which admit objects in the ontology as mere *nodes* or *bearers* of the relations – leave a number of questions open: where do exactly structures cease to exist and objects begin to? Are these objects too *thin* to be introduced in the ontology? More broadly, are *Priority-based OSR* and *Moderate OSR* really distinguished from the more radical *Eliminative OSR*, according to which *structure is all there is*?

Similar worries arise in the mathematical framework, where two main routes have been taken to resist the identity problem: first, one can adopt a weaker version of PII to distinguish objects in structures with non-trivial automorphism (Ladyman, 2005) – along the lines of Saunder's (2003) proposal in OSR – arguing that structurally indiscernible objects are distinguishable in virtue of the symmetric but irreflexive relations holding between them (i.e. ‘being the additive inverse of’ for relative and complex numbers). Still, this strategy has been criticized by MacBride (2006): symmetric and irreflexive relations actually presuppose the numerical diversity of objects, rather than grounding it. Second, the identity of objects can be understood as a *primitive* fact. This option, though, brings about a sort of 'dismissive attitude' (Parson, 2008) which runs into further difficulties, concerning how we can have an epistemic access to these primitive identity facts and how we can characterize objects whose identity is primitively defined.

On this basis, I endorsed an alternative approach, investigating the metaphysical claims presupposed by both OSR and *ante rem* structuralism. To do so, I started from the standard metaphysical picture suggested by Metaphysical Foundationalism (MF), in which the concepts of *priority* and *fundamentality* play a key role. From this perspective, I discussed ontological dependence, grounding and their different interpretations as a promising metaphysical toolkit to carve up the structure of reality and to elucidate the structuralist idea – bringing together scientific OSR and mathematical *ante rem* structuralism – that objects are *secondary* or *derivative* on structures. Metaphysical grounding, in particular, turned out to be a more promising tool, which provides a deeper insight into the structure of reality. That is because grounding has a stricter connection with metaphysical explanation: if x grounds y, then x metaphysically explains (or helps explaining) y. The relation between grounding and explanation has been differently articulated. In the present discussion, I opted for a compromise view, according to which the relationship between grounding and metaphysical explanation is regimented by some principles (i.e. 'inheritance' and 'involvement' principles) which establish a robust link between the two notions without identifying them. Significantly, grounding has been recently reconsidered within a non-foundationalist perspective, illustrated in detail by Bliss and Priest (2018). Several positions – challenging the standard properties of grounding and yet being logically and metaphysically plausible – are in fact available as alternatives to MF. In this framework, I took into account Infinitism and Coherentism as the most notable non-foundationalist approaches and developed my own account of *Weak Structuralism* (WS). As I argued, WS aims at combining some ideas of MF with the explanatory advantages of non-foundationalist views which – by admitting symmetric or not well-founded grounding relations – provide us with more powerful explanatory tools. In particular, WS appears as a mixture of Coherentism and Metaphysical Foundationalism. WS directly accounts for the relationship between objects and structures and it is based on a peculiar interpretation of grounding, which I called *Mutual Grounding*. *Mutual Grounding* does not correspond to symmetric grounding, for it envisages a distinction between two directions in which the grounding relations go: an upward direction, from objects to structures, in which the *identity* dimension is at play and a downward direction, from structures to objects, involving the *existential* dimension. Acknowledging this distinction, *Mutual Grounding* can be split into two distinct claims holding at the same time: *Object Identity* (objects are grounded for their identity in the structures they belong to) and *Structure Existence* (structures are grounded in objects for their existence). In addition to these two claims, a third *Structure Identity* claim accounts for the identity of structures by referring to those higher, more abstract structures which are relevant for both OSR (symmetry groups of group-theory) and *ante rem* structuralism (isomorphism types of abstract structures).

WS has two main implications: first, a more substantial conception of objects – i.e. objects as *quasi-thin objects* – emerges from the combination of *Object Identity* and *Structure Existence.* Second, *Structure Identity* allows us to come up with a specific interpretation of the well-foundedness of grounding: one in which the relevant entities are neither finitely grounded – as standardly required by MF – nor infinitely descending without a termination – as it happens in Infinitism and Coherentism. In WS, objects and structures are *bounded from below*, where the interpretation of their lower bound varies in the two debates: symmetry groups for scientific structuralism and isomorphism types for mathematical structuralism. The idea of having a lower bound introduces a non-standard account of the well-foundedness of grounding, which I understood as a form of *weak-fundamentality* (W-F). More precisely, WS corresponds to the following combination of properties: mutuality (M), anti-transitivity (AT), anti-reflexivity (AR) and weak-fundamentality (W-F). So defined, WS appears to be a broad conceptual framework. My main purpose was to apply this position to both scientific and mathematical structuralism in order to introduce novel positions – Weak Scientific Structuralism (WSS) and Weak Mathematical Structuralism (WMS) respectively – which can be advantageous when it comes to deal with their main objections. In fact, both WSS and WMS are advanced as middle ground positions which attempts to overcome some difficulties of OSR and *ante rem* structuralism (i.e. the 'relation without *relata*' objection and the identity problem) without abandoning their main intuition (i.e. the priority of structures).

Before putting WS at work in the scientific and the mathematical domain, it was useful to investigate the metaphysical claims OSR and *ante rem* structuralism are committed to. Concerning OSR, I returned to the main OSR-views – *Eliminative OSR*, *Priority-based OSR* and *Moderate OSR*

– and I illustrated their main metaphysical theses (*Fundamentality Thesis*, *Priority Thesis* and suitable modifications of them). I then considered their interpretation in terms of different forms of dependence (French, 2010) arguing – along the lines of Wolff (2011) – that dependence naturally favours non-eliminative conceptions of OSR. *Priority-based* and *Moderate OSR*, despite suggesting a non-eliminative approach towards objects, are still subject to variations of the 'relation without *relata*' objection and thus leave room for Weak Scientific Structuralism (WSS) as a novel interpretation of OSR – which is further motivated by the explanatory advantages of a grounding-based version of OSR.

WSS applies to quantum particles and quantum entanglement structures, which are posited in a new relationship of fundamentality in accordance with *Mutual Grounding*. On this view, quantum particles turned out to be *quasi-thin physical objects* endowed with both primary structural properties (the properties that remain invariant under symmetry-groups transformations, such as position and *momentum*) and secondary non-structural properties (*kind properties* such as spin and charge, which are hardly amenable to a structural characterization). On this basis, I distinguished quasi-thin objects from thin objects in *Priority-based* and *Moderate-OSR*, which are entirely defined by their structural properties. WSS has several benefits: first, the conjunction of *Object Identity* and *Structure Existence* provides a way to distinguish concrete structures from abstract structures – whose identification is another standard problem of OSR: physical structures are grounded for their (physical) existence in the (spatio-temporal) objects constituting them (*Structure Existence*). At the same time, we should not renounce the standard structuralist claim that quantum particles are grounded for their identity in quantum entanglement structures (*Object Identity*). Second, *Structure Identity* individuates physical structures by means of mathematical structures, i.e. symmetry-groups of group-theory, appealing to the notion of *shared structure* (Landry, 2007) which vindicates the prominent role played by group-theory in QM. Third, quasi-thin physical objects (as defined by definitions 1.A and 2.A) appear to be substantial enough to be legitimate *relata* of structures – as opposed to the very thin objects of *Priority-based OSR* and *Moderate OSR* – thus responding to the 'relation without *relata*' objection. However, quasi-thin physical objects are also weak enough to fit well with a structuralist framework, and also with the idea that higher structures (i.e. symmetry-groups) are ultimately prior to objects, as stated by *Structure Identity*.

A similar path has been explored in the mathematical framework; first I examined the metaphysical assumptions of *ante rem* structuralism and its interpretations in terms of dependence (Linnebo, 2008) and grounding (Wigglesworth, 2018). Wigglesworth's account of grounding appeared to be more suitable for understanding *ante rem* structuralism, for it is consistent with an *ante rem* individuation of structures. Second, I developed my own account of Weak Mathematical Structuralism (WMS) – whose articulation mirrors WSS – as a more moderate interpretation of *ante rem* structuralism, which in this case applies to numbers and the abstract structures they belong to. WMS aims at advocating a possible strategy to avoid the identity problem, distinguished from those already proposed in the literature. WMS's response relies on a more detailed investigation of the structural and non-structural properties of objects. In particular, I argued that if structural properties are interpreted in terms of an invariance account (Linnebo, 2008), then several counter-examples emerge. *Kind properties* proved particularly useful to introduce numbers as quasi-thin mathematical objects and to distinguish them from Shapiro's entirely structural objects. Such properties have been firstly explored in scientific structuralism; in the mathematical framework, they turn out to be related to counting and measurement facts, highlighting the different applicative uses of the natural, relative, rational, etc., numbers. As far as graphs are concerned, kind properties highlighted a specific feature of them, i.e. that of providing fundamental schematic representations which, in some cases, are essential to an understanding of relations in the physical world – which makes graphs particularly suitable to empirical applications. This has led to the formulation of WMS in terms of *Mutual Grounding*, which involves two different grounding claims holding at the same time: *Object Identity* and *Structure Existence*.

Assuming these claims, *quasi-thin mathematical objects* have been more specifically set out (definitions 1.B and 2.B) and a possible solution to the identity problem has been submitted, by referring to the cases of the 2-elements unlabelled graph with no edges and + 1 and – 1 in the relative number structure – with a possible suggestion of how to apply a similar strategy to + i and – i in the complex numbers structure. This strategy is just apparently questioned by the problem of cross-structural identities in *ante rem* structuralism, since the focus on the relevance of context helps tackling this issue consistently with the main features of WMS. At the same time, an *ante rem* individuation of structures has been defended, as shown by the third *Structure Identity* claim. *Structure Identity* plays a twofold role: firstly, it retains the priority of structures, in accordance with an *ante rem* framework, by individuating them with reference to their isomorphism types – where no systems are at play. Secondly, it provides a bound from below for *Mutual Grounding*, thus preserving WMS from typical objections of circularity.

On the whole, WSS and WMS fulfill the main metaphysical properties of WS broadly understood (M, AT, AR, W-F) and entail a new link between *grounding*, *fundamentality* and *priority.*

This link has been reformulated by claiming that both views involve a combination of the *Parity Thesis* (physical/mathematical objects and physical/mathematical structures are equi-fundamental) and the *Priority Thesis* (symmetry groups/isomorphism types are prior to physical/mathematical structures and to physical/mathematical objects in the sense of being their lower bound).

Moreover, both WSS and WMS invoke a different approach to metaphysical explanation, i.e. a form of *holistic explanation* which goes hand in hand with the relation of *Mutual Grounding*: structures explain objects for their identity and objects explain structures for their existence. On these grounds, WS – along with its specific applications, i.e. WSS and WMS – deserves to be taken seriously along with the other non-foundationalist views discussed by Bliss and Priest (2018). In fact, its unique combination of features appears to be not only logically plausible (as opposed to non anti-symmetry, the *mutuality* of WS is in principle consistent with anti-reflexivity) but also meta-physically and explanatorily advantageous.

References

Ainsworth, M. (2010). What is Ontic Structural Realism? *Studies in History and Philosophy of Modern Physics,* 41, 50–57.

Allaire, E. (1963). Bare Particulars. *Philosophical Studies,* 14, 1–8.

Antonelli, A. (1998). Definitions. In Craig (Ed.), *Routledge Encyclopedia, Logic & mathematics.* London: Routledge, 150–154.

Audi, P. (2012). Grounding: Toward a Theory of the In-Virtue-of Relation. *Journal of Philosophy*, 109, 685–711.

Awodey, S. (1996). Structure in Mathematics and Logic: A Categorical Perspective. *Philosophia Mathematica*, 4(3), 209–237.

Bain, J. (2009). Motivating Structural Realist Interpretations of Spacetime. Manuscript.

Benacerraf, P. (1965). What Numbers Could not Be. *Philosophical Review*, 74, 47–73.

Benacerraf, P. (1973). Mathematical Truth. *The Journal of Philosophy*, 70, 661–79.

Bennett, K. (2017). *Making Things Up.* Oxford: Oxford University Press.

Bigaj, T., (2015a). Dissecting Weak Discernibility of Quanta. *Studies in History and Philosophy of Modern Physics*, 50, 43–53.

Bigaj, T. (2015b). On Discernibility and Symmetries. *Erkenntnis*, 80, 117–135.

Bliss, R. (2014). Viciousness and Circles of Ground. *Metaphilosophy*, 45(2), 245–256.

Bliss, R., Priest, G. (2018). *Reality and Its Structure, Essays of Fundamentality*. Oxford: Oxford University Press.

Bokulich, A., Bokulich, P. (2011). *Scientific Structuralism.* Dordrecht: Springer.

Branding, K. (2011). Structuralist Approaches to Physics. Objects, Models and Modality. In A. Bokulich and P. Bokulich (Ed.), *Scientific Structuralism.* Dordrecht: Springer, 43–67.

Burgess, J. (1999). Review of Stewart Shapiro (1997). *Notre Dame Journal of Formal Logic,* 40, 283–291.

Busch, J. (2003). What Structures Could not Be. *International Studies in the Philosophy of Science*, 17, 211–225.

Button, T. (2006). Realistic Structuralism's Identity Crisis: A Hybrid Solution. *Analysis*, 66, 216–222.

Cameron, R. P. (2008). Truthmakers and Ontological Commitment. *Philosophical Studies* 1(140), 1–18.

Cantor, G. (1883). *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*. Leipzig: Teubner.

Cao, T. (2003). Can We Dissolve Physical Entities into Mathematical Structures? In Symonds (Ed.), *Special Issue: Structural Realism and Quantum Field Theory*, *Synthese*, 136(1), 57–71.

Cassirer, E. ([1923] 1953). *Substance and Function and Einstein's Theory of Relativity*. New York: Dover Press.

Castellani, E. (1998). Galilean Particles: An Example of Constitution of Objects. In Castellani (Ed.), *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*. Princeton: Princeton University Press, 181–94.

Chakravartty, A. (1998). Semirealism. *Studies in History and Philosophy of Modern Science*, 29, 391–408.

Chakravartty, A. (2003). The Structuralist Conception of Objects. *Philosophy of Science*, 70, 867–878.

Chihara, C. S. (2004). *A Structural Account of Mathematics*. Oxford: Oxford University Press.

Correia, F., Schnieder, B. (2012). *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge: Cambridge University Press.

Correia, F. (2017). Real Definitions. *Philosophical Issues*, 27 (1), 52–73.

Correia, F., and Skiles, A. (2019). Grounding, Essence and Identity. *Philosophy and Phenomenological Research* (3), 642-670.

Cotnoir, A. J., Bacon, A. (2012). Non-Well Founded Mereology. *The Review of Symbolic Logic*, 5(2), 187–204.

Caulton, A. (2013). Discerning 'Indistinguishable' Quantum Systems. *Philosophy of Science*, 80, 49–72.

Dasgupta, S. (2014). On the Plurality of Grounds. *Philosopher's Imprint*, 14 (20), 1–28.

Dedekind, R. (1872). *Stetigkeit und Irrationale Zahlen.* Translated as "Continuity and Irrational Numbers", in W. W. Beman (Ed.), *Essays on the Theory of Numbers.* New York: Dover Press, 1963, 1–27.

Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg. Translated as "The Nature and Meaning of Numbers", in Woodruff Beman (Ed.), *Essays on the Theory of Numbers.* Chicago: Open Court, 1901, pp. 29–115.

Demopoulos, W. and Friedman, M. (1985). Critical Notice: Bertrand Russell's The Analysis of Matter: Its Historical Context and Contemporary Interest. *Philosophy of Science*, 52, 621–639.

Dieks, D., Versteegh, M. (2008). Identical Quantum Particles and Weak Discernibility. *Foundations of Physics*, 38, 923–934.

Dipert , R. R. (1997). The Mathematical Structure of the World: the World as Graph. *Journal of Philosophy*, 94, 329–358.

Dixon, T. S. (2016) What Is the Well-Foundedness of Grounding? *Mind*, 125(498), 439–468.

Dorato, M. (2000). Substantivalism, Relationalism and Structural Spacetime Realism. *Foundations of Physics*, 30(10), 1605–28.

Esfeld, M. (2004). Quantum Entanglement and a Metaphysics of Relations. *Studies in the History of Philosophy of Physics*, 35B, 601–617.

Esfeld, M. and Lam, V. (2008). Moderate Structural Realism about Space-Time. *Synthese*, 160, 27–46.

Esfeld, M. and Lam, V. (2011). Ontic Structural Realism as a Metaphysics of Objects. In A. Bokulich and P. Bokulich (Ed.), *Scientific structuralism.* Dordrecht: Springer, 143–160.

Fine, A. (1984). The Natural Ontological Attitude. In Leplin (Ed.), *Scientific Realism.* University of California Press, 83–107.

Fine, K. (1994a). Essence and Modality. *Philosophical Perspectives,* 8 (1), 1–16.

Fine, K. (1994b). Ontological Dependence. *Proceedings of the Aristotelian Society*, 95, 269–9.

Fine, K. (1995). The Logic of Essence. *Journal of Philosophical Logic*, 24(3), 241–273.

Fine, K. (2001). The Question of Realism. *Philosophers' Imprint*, 1(2), 1–30.

Fine, K. (2010). Some Puzzles of Ground. *Notre Dame Journal of Formal Logic*, 51(1), 97–118.

Fine, K. (2012). Guide to Ground. In Correia and Schnieder (Ed.), *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge: Cambridge University Press, 37–80.

French, S. (1989). Identity and Individuality in Classical and Quantum Physics. *Australasian Journal of Philosophy*, 67, 432–446.

French, S. (1998). On the Withering Away of Physical Objects. In Branding and Castellani (Ed.), *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*. Princeton: Princeton University Press, 93–113.

French, S. (1999). Models and Mathematics in Physics: the Role of Group Theory. In Butterfield and Pagonis (Ed.), *From Physics to Philosophy.* Cambridge: Cambridge University Press, 187–207.

French, S. (2006). Structure as a Weapon of the Realist. *Proceedings of the Aristotelian Society*, 106, 1–19.

French, S. (2010). The Interdependence of Structure, Objects and Dependence. *Synthese,* 175, 89–109.

French, S. (2014). *The Structure of the World: Metaphysics and Representation*. Oxford: Oxford University Press.

French, S. (2019). Identity and Individuality in Quantum Theory. In E.N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL = <https://plato.stanford.edu/archives/win2019/entries/qt-idind/>.

French, S. and Ladyman, J. (2003a). Remodelling Structural Realism: Quantum Physics and the Metaphysics of Structure. *Synthese*, 136, 31–56

French, S., Ladyman, J. (2003b). Between Platonism and Phenomenalism: Reply to Cao. *Synthese*, 136, 73–78.

French, S., Redhead, M. (1988). Quantum Physics and the Identity of Indiscernibles. *British Journal for the Philosophy of Science*, 39 (2), 233–46.

French, S., Krause, D. (2006). *Identity in Physics: a Formal, Historical and Philosophical Approach.* Oxford: Oxford University Press.

Frigg, R. and Votsis, I. (2011). Everything You Always Wanted to Know About Structural Realism but Were Afraid to Ask. *European Journal for Philosophy of Science*, 1(2), 227–276.

Giere, R. (1985). Constructive Realism. In Churchland and Hooker (Ed.), *Images of Science*. Chicago: Chicago University Press, 75–98.

Gödel, K. (1944). Russell's Mathematical Logic. In Benacerraf and Putnam (Ed.), P*hilosophy of Mathematics: Selected Readings.* Cambridge: Cambridge University Press (second edition), 1983, 447–469.

Goldman, A. (1967). A Causal Theory of Knowing. *Journal of Philosophy*, 64 (12), 357–372.

Grassman, H. (1844). *Die Lineale Ausdehnungslehre*, Leipzig: Wiegand. Translated as *A New Branch of Mathematics,* Kannenberg (trans.). Chicago: Open Court, 1995.

Grassman, H. (1972). *Gessammelte Mathematische und Physicalische Werke 1*. In Engels (Ed.), New York: Johnson Reprint Corporation.

Hale, B. (1996). Structuralism's Unpaid Epistemological Debts. *Philosophia Mathematica*, 4(2), 124–147.

Hale, B., Wright, C. (2001). Implicit Definition and the A Priori. In Boghossian and Peacocke (Ed.), *New Essays on the A Priori.* Oxford: Clarendon Press, 286–319.

Hellman, G. (1989). *Mathematics Without Numbers: Towards a Modal Structural Interpretation*, Oxford: Oxford University Press.

Hellman, G. (1996). Structuralism Without Structures. *Philosophia Mathematica*, 4(2), 100–123.

Hellman, G. (2001). Three Varieties of Mathematical Structuralism. *Philosophia Mathematica*, 9 (3), 184–211.

Hellman, G., Shapiro, S. (2018). *Mathematical Structuralism.* Cambridge: Cambridge University Press.

Hilbert , D. (1899). *Grundlagen der Geometrie*. Leipzig: Teubner. Translated as *Foundations of Geometry*, Townsend (trans.). La Salle: Open Court, 1959.

Huggett, N. and Norton, J. (2014). Weak Discernibility for Quanta, The Right Way. *British Journal for the Philosophy of Science*, 65, 39–58.

Keränen, J. (2001). The Identity Problem for Realist Structuralism. *Philosophia Mathematica* (III) 9, 308-330.

Keränen, J. (2006). The Identity Problem for Realist Structuralism II: a Reply to Shapiro. In Fraser MacBride (Ed.), *Identity and Modality.* Oxford: Oxford University Press, 34–69.

Ketland, J. (2006). Structuralism and the Identity of Indiscernibles. *Analysis*, 66 (4), 303–15.

Kitcher, P. (1983). *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Koslicki, K. (2012). Varieties of Ontological Dependence. In Correia and Schnieder (Ed.), *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge: Cambridge University Press, 186–213.

Ladyman, J. (1998). What is Structural Realism? *Studies in History and Philosophy of Science*, 29, 409–424.

Ladyman, J. (2005). Mathematical Structuralism and the Identity of Indiscernibles. *Analysis*, 65, 218–221.

Ladyman (2007). On the Identity and Diversity of Individuals. *The Proceedings of the Aristotelian Society*, Supplementary Volume, 81, 23–43.

Ladyman, J. (2020). Structural Realism. In E. N Zalta (Ed.), *Stanford Encyclopedia of Philosophy,* URL = <https://plato.stanford.edu/archives/spr2020/entries/structural-realism/>.

Ladyman, J., Ross, D. (2007). *Every Thing Must Go: Metaphysics Naturalised.* Oxford: Oxford University Press.

Ladyman, J., Leitgeb, L. (2008). Criteria of Identity and Structuralist Ontology. *Philosophia Mathematica* 16, 388–396.

Ladyman, J., Bigaj, T. F. (2010). The Principle of the Identity of Indiscernibles and Quantum Mechanics. *Philosophy of Science*, 77, 117–136.

Landry, E. (2007). Shared Structure Need not be Shared Set-Structure. *Synthese*, 158, 1–17

Laudan, L. (1981). A Confutation of Convergent Realism. *Philosophy of Science*, 48, 19–49.

Leitgeb, H. (2020) On Non-Eliminative Structuralism: Unlabeled Graphs as a Case Study: Part B. *Philosophia Mathematica*, https://doi.org/10.1093/philmat/nkaa009.

Lewis, D. (1986). *On the Plurality of Worlds.* Oxford: Blackwell.

Linnebo, Ø. (2003). Critical Notice of Shapiro (Philosophy of Mathematics). *Philosophia Mathematica*, 11(2), 92–104.

Linnebo, Ø. (2008). Structuralism and the Notion of Dependence. *Philosophical Quarterly*, 58 (230), 381–398.

Linnebo, Ø. (2013). The Potential Hierarchy of Sets. *Review of Symbolic Logic*, 6(2), 205–228.

Linnebo, Ø., Pettigrew (2014). Two Types of Abstraction for Structuralism. *The Philosophical Quarterly*, 64(255), 267–283.

Linnebo, Ø. (2017). Predicative and Impredicative Definitions. *The Internet Encyclopedia of Philosophy,* URL = https://www.iep.utm.edu/predicat/.

Linnebo, Ø. (2018). *Thin Objects, an Abstractionist Account.* Oxford: Oxford University Press.

Lowe, E. J. (1989). What Is a Criterion of Identity? *The Philosophical Quarterly*, 39(154), 1–21.

Lowe, E. J. (1994). Ontological Dependency. *Philosophical Papers*, 23(1), 31–48.

Lowe, E. J. (2003). Individuation. In Loux and Zimmerman (Ed.), *Oxford Handbook of Metaphysics.* Oxford: Oxford University Press, 75–95.

Lowe, E. J. (2005) [2010]. Ontological Dependence. In E.N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL = <https://plato.stanford.edu/archives/spr2010/entries/dependence-ontological/>.

Lowe, E. J. (2008). Two Notions of Being: Entity and Essence. *Royal Institute of Philosophy Supplement*, 62, 23–48.

Lowe, E. J. (2012). Asymmetrical Dependence in Individuation. In Correia and Schnieder (Ed.), *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge: Cambridge University Press, 214–233.

Lowe, E. J. (2013). Some Varieties of Metaphysical Dependence. In Schnieder, Hoeltje and Steinberg (Ed.), *Dependence (Basic Philosophical Concepts).* Munich: Philosophia Verlag, 193–210.

MacBride, F. (2005). Structuralism Reconsidered. In Shapiro (Ed.), *Oxford Handbook of Philosophy of Mathematics and Logic.* Oxford: Clarendon, Press, 563–589.

MacBride, F. (2006). What Constitutes the Numerical Diversity of Mathematical Objects? *Analysis*, 66 (1), 63–69.

Massimi, M. (2011). Structural Realism: A Neo-Kantial Perspective. In A. Bokulich, P. Bokulich (Ed.) *Scientific Structuralism.* Dordrecht: Springer, 1–23.

Maurin (2018). Grounding and Metaphysical Explanation: It's Complicated. *Philosophical Studies*, 176, 1573–1594.

Maxwell (1970). Structural Realism and the Meaning of Theoretical Terms. In Winokur and Radner (Ed.), *Analyses of Theories, and Methods of Physics and Psychology: Minnesota Studies in the Philosophy of Science*, Vol. IV. Minneapolis: University of Minnesota Press, 181–192.

McLaughlin, B., Bennett, K. (2018). Supervenience. In E.N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL = https://plato.stanford.edu/archives/win2018/entries/supervenience/.

McKenzie, K. (2014). Priority and Particle Physics: Ontic Structural Realism as a Fundamentality Thesis. *British Journal for the Philosophy of Science*, 65(2), 353– 80.

McKenzie, K. (forthcoming). Structuralism in the Idiom of Determination. *British Journal for the Philosophy of Science*: axx061.

Menzel, C. (2018). Haecceities and Mathematical Structuralism. *Philosophia Mathematica* (III) 2, 84–111.

Moore, G. E. (1900). Identity. *Proceedings of the Aristotelian Society*, 1, 103–27.

Morganti, M. (2004). On the Preferability of Epistemic Structural Realism. *Synthese*, 142, 81–107.

Morganti, M. (2012). Identity in Physics: Properties, Statistics and the (Non-) Individuality of Quantum Particles. In Henk W. de Regt (Ed.), *Epsa Philosophy of Science: Amsterdam 2009*. Springer, 227–237.

Morganti, M. (2018). The Structure of Physical Reality: Beyond Foundationalism. In Bliss and Priest (Ed.), *Reality and Its Structure, Essays of Fundamentality*. Oxford: Oxford University Press, 254–272.

Muller, F., Saunders, S. (2008). Discerning Fermions. *British Journal for the Philosophy of Science* 59, 499–548.

Newman, P. (1928). Mr. Russell's Causal Theory of Perception. *Mind*, 37, 137–148.

Nodelman, U., Zalta, E. N. (2014). Foundations for Mathematical Structuralism. *Mind*, 123(489), 39–78.

Norton, J. (2015). Weak Discernibility and Relations between Quanta. *Philosophy of Science*, 82, 1188–1199.

O'Conaill, D. (2014). Ontic Structural Realism and Concrete Objects. *The Philosophical Quarterly*, 64 (255), 284–300.

Parsons, C. (1990). The Structuralist View of Mathematical Objects. *Synthese*, 84, 303–346.

Parsons, C. (2004). Structuralism and Metaphysics. *Philosophical Quarterly*, 54, 56–7.

Parsons, C. (2008). *Mathematical Thought and its Objects.* Cambridge: Cambridge University Press.

Pincock, C. (2007). A Role for Mathematics in the Physical Sciences. *Nôus,* 41(2), 253–275.

Poincaré, H. ([1905] 1952). *Science and Hypothesis*. New York: Dover Press.

Poincaré, H. (1906). Les Mathematiques et la Logique. *Revue de Métaphysique et de Morale*, 14, 294–317. Translated as "Mathematics and Logic", II. In Ewald (Ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics* (volume 2). Oxford: Oxford University Press, 1996, 1038-10.

Poincarè, H. (1908). *Science et Methode*, Paris: Flammarion. Translated in *The Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method,* Halsted (trans.). New York: The Science Press, 1921, 359–546.

Psillos, S. (1995). Is Structural Realism the Best of Both Worlds? *Dialectica*, 49, 15–46.

Psillos, S. (1999). *Scientific Realism: How Science Tracks Truth*. London: Routledge.

Psillos, S. (2001). Is Structural Realism Possible? *Philosophy of Science*, 68 (Supplementary Volume), S13–S24.

Psillos, S. (2006). The Structure, the Whole Structure and Nothing but the Structure. *Philosophy of Science. Proceedings*, 73, 560–570.

Putnam, H. (1967). Mathematics without Foundations. *The Journal of Philosophy*, 64(1), 5–22.

Putnam, H. (1975). *Mathematics, Matter and Method.* Cambridge: Cambridge University Press.

Quine, W. V. (1960). *Word and Object.* Harvard: Harvard University Press.

Quine, W. V. (1976). Grades of Discriminability. *Journal of Philosophy,* 73, 113–16.

Rabin, G. O., Rabern, B. (2016). Well Founding Grounding Grounding. *Journal of Philosophical Logic*, 45(4), 349–379.

Raven, M. J. (2013). Is Ground a Strict Partial Order? *American Philosophical Quarterly*, 50(2), 191–199.

Raven, M. J. (2015). Ground. *Philosophy Compass*, 10 (5), 322–333.

Reck, E. (2003). Dedekind's Structuralism: an Interpretation and Partial Defense. *Synthese,* 137, 369–419.

Reck, E., Price, M. (2000). Structures and Structuralism in Contemporary Philosophy of Mathematics. *Synthese* 125, 341-38.

Reck, E., Schiemer, G. (2020). Structuralism in the Philosophy of Mathematics. In E.N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL= <https://plato.stanford.edu/archives/spr2020/entries/structuralism-mathematics/>.

Redhead, M. (2001). The Intelligibility of the Universe. In O'Hear (Ed.), *Philosophy at the New Millennium.* Cambridge: Cambridge University Press, 73–90.

Resnik, M. D. (1997). *Mathematics as a Science of Patterns.* Oxford: Oxford University Press.

Resnik, M. D. (1981). Mathematics as a Science of Patterns: Ontology and Reference. *Noûs*, 15(4), 529–550.

Resnik, M. D. (1988). Mathematics from the Structural Point of View. *Revue Internationale de Philosophie*, 42(167), 400–424.

Roberts, B. W. (2011). Group Structural Realism. *British Journal for the Philosophy of Science*, 62, 47–69.

Rosen, G. (2010). Metaphysical Dependence: Grounding and Reduction. In Hale and Hoffmann (Ed.), *Modality: Metaphysics, Logic, and Epistemology.* Oxford: Oxford University Press, 109–36.

Russell, B. (1903). *The Principles of Mathematics.* Cambridge: Cambridge University Press.

Russell, B. (1908). Mathematical Logic as Based on a Theory of Types. *American Journal of Mathematics*, 30, 222-262.

Russell, B. ([1912] 1959). *The Problems of Philosophy*. Oxford: Oxford University Press.

Russell, B. (1927). *The Analysis of Matter.* New York: Harcourt, Brace & Co.

Saunders, S. (2003). Physics and Leibniz's Principles. In Branding, Castellani (Ed.), *Symmetries in Physics: Philosophical Reflection.* Cambridge: Cambridge University Press, 289–307.

Schaffer, J. (2009). On What Grounds What. In Chalmers, Manley, and Wasserman (Ed.), *Metametaphysics: New Essays on the Foundations of Ontology*. Oxford: Oxford University Press, 347–283.

Schaffer, J. (2012). Grounding, Transitivity, and Contrastivity. In Correia and Schnieder (Ed.), *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge: Cambridge University Press, 122–138.

Schiemer, G., Korbmacher, J. (2017). What are Structural Properties? *Philosophia Mathematica* (III) 26, 295–323.

Schiemer, G., Wigglesworth, J. (2019). The Structuralist Thesis Reconsidered. *British Journal of Philosophy of Science*, 70, 1201–1226.

Shapiro, S. (1997). *Philosophy of Mathematics: Structure and Ontology.* Oxford: Oxford University Press.

Shapiro, S. (2000). *Thinking about Mathematics*. Oxford: Oxford University Press.

Shapiro, S. (2006a). Structure and Identity. In Fraser MacBride (Ed.), *Identity and Modality.* Oxford: Oxford University Press, 34–69.

Shapiro (2006b). The Governance of Identity. In MacBride (Ed.), *Identity and Modality.* Oxford: Oxford University Press, 164–173.

Shapiro, S. (2008). Identity, Indiscernibility, and Ante Rem Structuralism: The Tale of i and –i. *Philosophia Mathematica* (III), 16, 285–30.

Sider, T. (2012). *Writing the Book of the World*. Oxford: Oxford University Press.

Stachel, J. (2002). The Relations Between Things versus the Things between Relations. The Deeper Meaning of the Hole Argument. In Malament (Ed.), *Reading Natural Philosophy: Essays in the History and Philosophy of Science and Mathematics.* Chicago and LaSalle: Open Court, 231–266.

Stanford, P. K. (2003). No Refuge for Realism: Selective Confirmation and the History of Science. *Philosophy of Science*, 70, 913–925.

Stein, H. (1989). Yes, but… Some Skeptical Remarks on Realism and Antirealism. *Dialectica*, 43, 47–65.

Tahko, T., Lowe, E.J. (2020). Ontological dependence. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL = https://plato.stanford.edu/archives/win2016/entries/dependence-ontological/>.

Tahko, T. (2015). *An Introduction to Metametaphysics.* Cambridge: Cambridge University Press.

Tahko, T. (2018). Fundamentality. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy*, URL = <https://plato.stanford.edu/archives/fall2018/entries/fundamentality/>.

Tahko, T. and Lowe, E. J. (2020). Ontological Dependence. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy,* URL = <https://plato.stanford.edu/archives/fall2020/entries/dependence-ontological/>.

Teller, P. (1989). Relativity, Relational Holism, and the Bell Inequalities. In Cushing and McMullin (Ed.), *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem.* Notre Dame: University of Notre Dame Press, 208–221.

Thompson, N. (2018). Metaphysical Interdependence, Epistemic Coherentism, and Holistic Explanation. In Bliss and Priest (Ed.), *Reality and Its Structure, Essays of Fundamentality*. Oxford: Oxford University Press, 107–126.

Thompson, N. (2019). Questions and Answers. Metaphysical Explanation and the Structure of Reality. *Journal of the American Philosophical Association*, 5 (1), 98–116.

Trogdon (2012). An Introduction to Grounding. In Hoeltje, Schnieder, and Steinberg (Ed.), *Varieties of Dependence: Ontological Dependence, Grounding, Supervenience, Response-Dependence (Basic Philosophical Concepts).* Munich: Philosophia Verlag, 97–122.

Votsis, I. (2005). The Upward Path to Structural Realism. *Philosophy of Science*, 72, 1361–1372.

Weyl, H. (1931). *The Theory of Groups and Quantum Mechanics,* Robertson (trans.). New York: Dover Press, 1950.

Weyl, H. (1918). *Das Kontinuum*. Leipzig: Verlag von Veit & Comp. Translated as *The Continuum*, Pollard and Bole (trans.). New York: Dover Press, 1994.

Wigglesworth, J. (2018). Grounding in Mathematical Structuralism. In Bliss and Priest (Ed.), *Reality and its Structure: Essays in Fundamentality.* Oxford: Oxford University Press, 217–236.

Wigner, E. (1939). On Unitary Representations of the Inhomogeneous Lorentz Group. *The Annals of Mathematics, Second Series*, 40, 149–204.

Wolff, J. (2011). Do Objects Depend on Structures? *British Journal for the Philosophy of Science*, 63 (3), https://doi.org/10.1093/bjps/axr041.

Worrall, J. (1989). Structural Realism: the Best of Both Worlds? *Dialectica,* 43, 99–124.